A SURVEY ON LAGRANGIAN SUBMANIFOLDS IN CONSERVATIVE DYNAMICS: FROM K.A.M. TO WEAK K.A.M. VARIÉTÉS LAGRANGIENNES EN DYNAMIQUE CONSERVATIVES: DE K.A.M. À K.A.M. FAIBLE

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Abstract. These are notes of a talk that I gave at the first congress of the French Mathematical Society in june 2016.
The setting is the one of Hamiltonian dynamics. More precisely, we introduce the classical notion of Lagrangian submanifold for completely integrable systems and then for their perturbations via the K.A.M. theorem. Then, focusing on the so-called Tonelli Hamiltonians, we explain the link between Hamiltonian dynamics and Lagrangian variational methods. This is the main tool to introduce an Aubry-Mather theory that gives invariant sets that are the phantoms of the invariant Lagrangian submanifolds when they disappear. We will conclude by some results on the link between the dynamics and the shape of the invariant minimizing sets.
Many thanks to the SMF for having organized this great congress.

Résumé. *Ce texte est celui d'une conférence que j'ai donné lors du premier congrès de la société mathématique de France en juin 2016.
On travaille en dynamique hamiltonienne. On s'intéresse aux variétés lagrangiennes invariantes qui apparaissent de manière classique dans les systèmes complètement intégrables, et à leurs perturbations via la théorie K.A.M. Se plaçant dans le cadre des hamiltoniens de Tonelli, on explique le lien entre dynamique hamiltonienne et méthodes variationnelles lagrangiennes. C'est l'outil clé pour introduire une théorie d'Aubry-Mather qui produit des ensembles invariants, fantômes des variétés lagrangiennes quand elles n'existent plus. On conclura par quelques remarques sur les liens entre dynamique et la forme des ensembles invariants minimisants.
Je tiens à remercier la SMF d'avoir organisé ce grand congrès.*

Key words: Hamilton Dynamics, Lagrangian submanifolds, Toneli Hamiltonians, Lagrangian variational methods, Aubry-Mather theory, K.A.M. theory, weak K.A.M. theory.

Mots clefs: Dynamique hamiltonienne, sous-variétés lagrangiennes, hamiltoniens de Tonelli, Méthodes variationnelles lagrangiennes, théorie d'Aubry-Mather, théorie K.A.M, théorie K.A.M. faible.

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1. Hamiltonian systems

When studying classical mechanics, we are often led to study the so-called *conservative systems* (see [8]), for example systems of massive points that move in a potential field.
If we consider three points with masses m_1, m_2, m_3 and positions q_1, q_2, q_3 in the space $\mathbb{R}^3$ which move in the field with potential energy U, their motion is given by the system of differential equations

$$m_i \ddot{q}_i = -\frac{\partial U}{\partial q_i}. \tag{1}$$

For example, the 3-body problem of celestial mechanics is such a system with

$$U(q_1, q_2, q_3) = -\frac{m_1.m_2}{\|q_1 - q_2\|} - \frac{m_3.m_2}{\|q_2 - q_3\|} - \frac{m_1.m_3}{\|q_1 - q_3\|}.$$

Observe that if we define on a subset of $(\mathbb{R}^3)^3 \times (\mathbb{R}^3)^3$ the function H by

$$H(q, p) = \frac{1}{2m_1}\|p_1\|^2 + \frac{1}{2m_2}\|p_2\|^2 + \frac{1}{2m_3}\|p_3\|^2 + U(q_1, q_2, q_3),$$

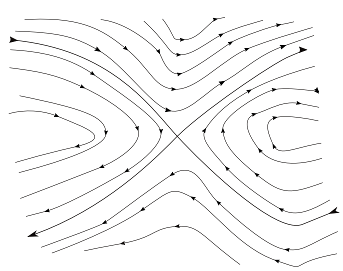
then $q : I^1 \to (\mathbb{R}^3)^3$ is a solution of Equation (1) if and only if

$$(q, p) = (q_1, q_2, q_3, m_1\dot{q}_1, m_2\dot{q}_2, m_3\dot{q}_3)$$

is solution of the *Hamilton equations*

$$\dot{q} = \frac{\partial H}{\partial p}(q, p) \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p). \tag{2}$$

More generally, a Hamiltonian system can be defined on any symplectic manifold (M, ω) (i.e. manifold endowed with a closed non-degenerate 2-form) when given a C^2 function $H : M \to \mathbb{R}$, which is called a *Hamiltonian*. In a good chart (a symplectic one), the equations of motion are just given by the system (2). The great advantage is that the equations have the form (2) in all symplectic charts.
The Hamiltonian flow[2] $(\varphi_t^H)_{t\in\mathbb{R}}$ describes the evolution of the system. It preserves the function H. Hence the level sets of H are preversed by the Hamiltonian flow of H.



The level sets of a Hamiltonian that is defined on $\mathbb{R}^2$

[1] I is an interval.

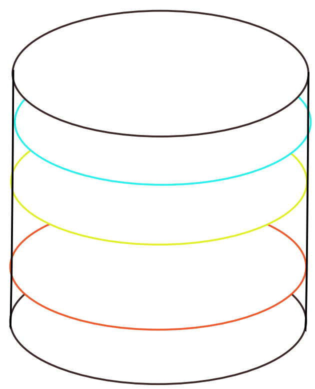
[2] We will always assume the flow is complete, i.e. that the solutions are defined on the whole $\mathbb{R}$.

The volume (that is the usual volume in the symplectic charts) is also preserved by any Hamiltonian flow. Hence there cannot exist any attractor or repellor for such a flow. But in general, there are many *recurrences* (points whose orbit come closer and closer to the original point) : this is a consequence of the so-called Poincaré recurrence theorem (see [8]).
It can be proved that a geodesic flow is Hamiltonian.

2. The completely integrable Hamiltonian systems

The baby example is the following one. Let us consider a Hamiltonian function $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ that only depends on the last n variables $H(q,p) = h(p)$. Then the Hamiltonian flow is given by

$$\forall (q,p) \in \mathbb{T}^n \times \mathbb{R}^n, \varphi_t^H(q,p) = (q + t\frac{\partial h}{\partial p}(p), p).$$



The foliation into invariant tori

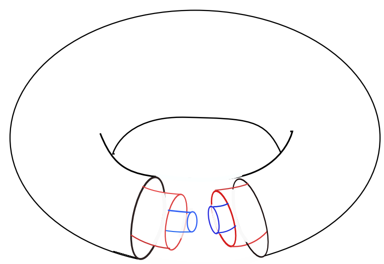
Observe that $(\varphi_t^H)_{t\in\mathbb{R}}$ has an invariant foliation into n-dimensional invariant tori $\mathbb{T}^n \times \{p_0\}$. Moreover, the restriction of the flow to each of these tori is the rotation flow $(q, p_0) \mapsto (q + t\alpha, p_0)$ with *rotation vector* $\alpha = \frac{\partial h}{\partial p}(p_0)$. In particular, all the orbits are bounded.
We can say a little more. Write $\alpha = (\alpha_1, \dots, \alpha_n)$. Let k be the dimension of the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ that is generated by $\alpha_1, \dots, \alpha_n$. Then $\mathbb{T}^n \times \{p_0\}$ is foliated by invariant k-dimensional tori such that the dynamics restricted to each of these low-dimensional tori is minimal (i.e. all the orbits are dense). For example

- if $\dim_{\mathbb{Q}}(\mathbb{Q}.\alpha_1 + \dots \mathbb{Q}.\alpha_n) = 1$, all the orbits in $\mathbb{T}^n \times \{p_0\}$ are periodic;
- if $\dim_{\mathbb{Q}}(\mathbb{Q}.\alpha_1 + \dots \mathbb{Q}.\alpha_n) = n$, all the orbits are dense in $\mathbb{T}^n \times \{p_0\}$.

More generally, in dimension $2n$, a Hamiltonian $H : M \to \mathbb{R}$ is *completely integrable* on a open subset U that is invariant by the flow $(\varphi_t^H)_{t\in\mathbb{R}}$ if there exist n C^2 functions $H_1, \dots, H_n : U \to \mathbb{R}$ such that

- the functions $H_1, \dots, H_n$ are constant along the orbits of the Hamiltonian flow of H;
- each function H_i is constant along each orbit of the Hamiltonian flow of H_j;
- at every $x \in U$, the linear forms $dH_1(x), \dots, dH_n(x)$ are linearly independent.

Arnol'd-Liouville theorem (see [8]) tells us that under these hypotheses, if a connected component $\mathcal{N}$ of a n-level $\{H_1 = c_1, \dots, H_n = c_n\}$ is compact, then there exists a neighbourhood $\mathcal{V}$ of $\mathcal{N}$ in U and a neighbourhood V of 0 in $\mathbb{R}^n$ such that the Hamiltonian flow of H restricted to $\mathcal{V}$ is conjugated to the flow of some Hamiltonian $h : \mathbb{T}^n \times V \to \mathbb{R}$ that only depends on $p \in V$.
In other words, doing some change of coordinates in a neighbourhood of $\mathcal{N}$, we obtain the baby flow that we described before; in particular, $\mathcal{V}$ is foliated into invariant n-dimensional tori and the dynamics restricted to each invariant torus is conjugated to a rotation flow. Hence, the orbit of a point that is close to $\mathcal{N}$ remains close to $\mathcal{N}$: we have stability.



The foliation into invariant tori

The geodesic flow on an ellipsoid is completely integrable in an open dense set. But in general, the Hamiltonian systems are not integrable (except when $n = 1$) and some chaotic phenomena can be exhibited, but this is not the topic of this paper.

It is usually thought that the N body problem (N massive points moving in their Newtonian field) is not completely integrable, but no proof exists when assuming that $H_1, \dots, H_n$ are smooth (here $n = 3N$ because we have N bodies in a 3-dimensional space). The main difficulty is that we always have local complete integrability (this is a corollary of the Hamiltonian flow box theorem, see [1]) and so proving non-integrabiity implies knowing the global dynamics. In [17], it is proved that the 3-body problem is not analytically integrable.

3. Invariant Lagrangian tori

The invariant tori $\mathcal{N}$ that appear in the completely integrable Hamiltonian systems are all *Lagrangian*, which means

- the symplectic form restricted to every tangent space is zero

$$\forall x \in \mathcal{N}, \forall v, w \in T_x\mathcal{N}, \omega(x)(v, w) = 0;$$

- the dimension of $\mathcal{N}$ is half of the dimension of the ambient manifold.

When $n = 1$, any curve is Lagrangian because the symplectic form is antisymmetric: $\omega(x)(v, v) = 0$. But in higher dimension, the set of Lagrangian submanifolds is small in the set of n-dimensional submanifold (i.e. has no interior).
However, if we prescribe the restricted Hamiltonian flow to some invariant submanifold, we can sometimes conclude that the submanifold is Lagrangian.

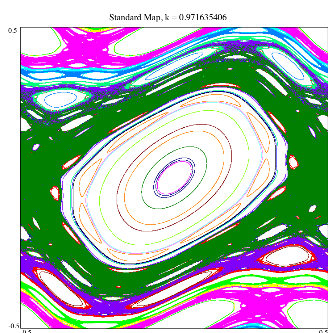
Proposition. *(M. Herman,* [14]*) In a $2n$-dimensional* exact symplectic manifold*, if a n-dimensional torus $\mathcal{N}$ is invariant by the time T of some Hamiltonian flow (φ_t) and such that $\varphi_{T|\mathcal{N}}$ is C^1-conjugate to a* minimal rotation*, then $\mathcal{N}$ is Lagrangian.*

Where

- the symplectic manifold (M, ω) is exact symplectic if the symplectic 2-form ω is exact: $\omega = d\lambda$; this is the case of $\mathbb{T}^n \times \mathbb{R}^n$;
- we recall that a map is minimal if all its orbits are dense.

Hence some restricted dynamics forces the submanifold to be Lagrangian. Similarily, the stable or unstable manifold of a hyperbolic fixed point is always Lagrangian.

When we perturb a completely Hamiltonian system, in general, the invariant foliation disappears: in particular, there is no invariant torus that is filled in by periodic orbits. But the K.A.M. theorem (for Kolmogorov-Arnol'd-Moser) tells us that a lot of invariant Lagrangian tori persist once a certain torsion hypothesis is satisfied (see [11]).



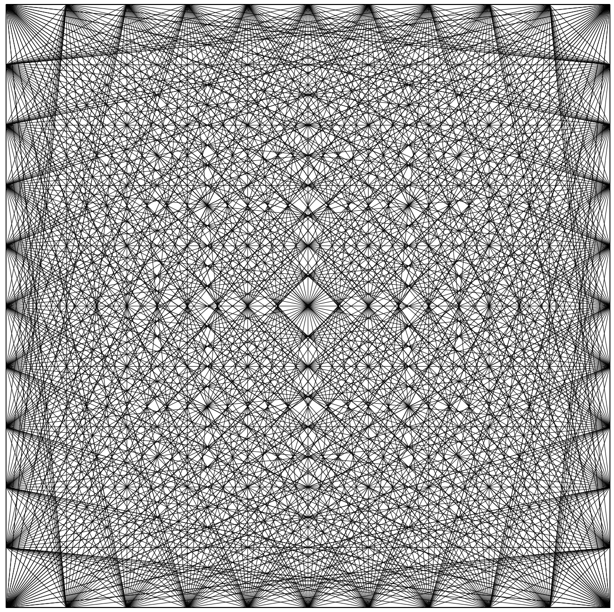
Invariant curves and chaotic regions

(Picture with Std Map 4.5 of James Meiss http://amath.colorado.edu/faculty/jdm/stdmap.html)

For the Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $H(q,p) = h(p)$, the torsion hypothesis is that $\det(\frac{\partial^2 h}{\partial p^2}) \neq 0$.

The persistent tori are those that correspond to rotation vectors $\alpha = (\alpha_1, \dots, \alpha_n) = \frac{\partial h}{\partial p}(p)$ that are *Diophantine*, i.e. such that there exist $\gamma > 0$ and $\tau > 0$ such that for every $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, every $j \in \mathbb{Z}$ so that $(k, j) \neq \mathbf{0}_{\mathbb{R}^{n+1}}$, then we have

$$|\alpha.k - j| \geq \frac{\gamma}{(j^2 + \sum_{i=1}^n k_i^2)^{\frac{\tau}{2}}}.$$



In black the rational lines

In other words, a vector is Diophantine if it is far from any rational hyperplane. It can be proved (see [9]) that the set of Diophantine vectors is

- small in the topological sense: it is a meager set;
- large in the measure sense: it has full Lebesgue measure.

Hence invariant Lagrangian submanifolds exist not only for completely integrable Hamiltonian systems but also for Hamiltonian systems that are C^∞-close to the completely integrable ones.

We have observed at the beginning of this section that a prescribed restricted dynamics implies that the invariant submanifold is Lagrangian. Now, we finish this section by noting that on a Lagrangian submanifold, the restricted dynamics can be anything. The interested readre can find a proof in [15].

Proposition. *Let $\mathcal{N}$ be a compact Lagrangian submanifold of M and let (f_t) be a flow on $\mathcal{N}$. Then there exists a Hamiltonian $H : M \to \mathbb{R}$ such that $\mathcal{N}$ is invariant by the Hamiltonian flow (φ_t) of H and such that $(\varphi_{t|\mathcal{N}}) = (f_t)$.*

Ideea of Proof. The idea of the proof is the following one, that we give only for a submanifold that is a n-dimensional torus and a flow that is smooth. Using a Weinstein tubular neighbourhood (see [18]), we can assume that $\mathcal{N}$ is $\mathbb{T}^n \times \{0\}$ in $\mathbb{T}^n \times \mathbb{R}^n$. Then we define a flow $(F_t)_{t\in\mathbb{R}}$ by $F_t(q,p) = (f_t(q), (Df_t(q))^{-T}p)$. It can be checked that it is Hamiltonian.

4. Tonelli Hamiltonians

Every cotangent bundle can be endowed with a symplectic form, but to be simpler we will only deal with $\mathbb{T}^n \times \mathbb{R}^n$. A *fiber* is then a subset $\{q\} \times \mathbb{R}^n$ where $q \in \mathbb{T}^n$. A *Tonelli Hamiltonian* is a C^2 function $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ that is

- convex in the fiber direction: the Hessian $\frac{\partial^2 H}{\partial p^2}(q,p)$ is positive definite;

- superlinear in the fiber direction: uniformly in $q \in \mathbb{T}^n$, we have

$$\lim_{\|p\| \to +\infty} \frac{H(q,p)}{\|p\|} = +\infty.$$

A geodesic flow is a Tonelli Hamiltonian flow. A mechanical system that is given by kinetic energy+potential energy is a Tonelli Hamiltonian.

We can associate a *Lagrangian* $L : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ to any Tonelli Hamiltonian H. The Lagrangian L is defined by

$$L(q,v) = \sup_{p \in \mathbb{R}^n} (p.v - H(q,p)).$$

In other word, L is Fenchel dual of H for the p variables. For such dual functions, it is known that there are some relations between the partial derivatives of the two functions. Then

- each $\frac{\partial L}{\partial v}(q,.) = \frac{\partial H}{\partial p}(q,.)^{-1}$ is a C^1 diffeomorphism of $\mathbb{R}^n$ and then
- $\mathcal{L} : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ defined by $\mathcal{L}(q,v) = (q, \frac{\partial L}{\partial v}(q,v))$ is a C^1 diffeomorphism ;
- the orbits of the Hamiltonian flow of H are the $\mathcal{L}(\gamma, \gamma')$ where, on every small time interval $[a,b]$, γ minimizes action function

$$\mathcal{F}(\eta) = \int_a^b L(\eta(t), \eta'(t))dt$$

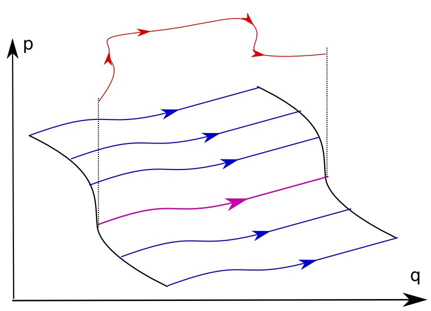
among the the C^1 arcs $\eta : [a,b] \to \mathbb{T}^n$ that have the same endpoints.

Remarks.

- In the case of a geodesic flow, this action can been seen as a length;
- in general, the Hamiltonian is defined on the cotangent bundle and the Lagrangian is defined on the tangent bundle. In the case of $\mathbb{T}^n$, the tangent and cotangent bundles are both $\mathbb{T}^n \times \mathbb{R}^n$.

Hence we obtain a variational formulation that allows us to find the orbits of a dynamical system: a small piece of orbit is a critical point (minimum) of the Lagrangian action. The existence of an invariant Lagrangian graph $\mathcal{N}$ implies a stronger result: the orbits in $\mathcal{N}$ are (locally) minimizing for every time interval.

Theorem. *(J. Mather) Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Tonelli Hamiltonian and let $\mathcal{N}$ be a Lagrangian submanifold that is an invariant graph by the Hamiltonian flow of H. Let $t \mapsto (q(t), p(t))$ be an orbit that is contained in $\mathcal{N}$. Then for every $[a, b] \subset \mathbb{R}$, the arc $q_{|[a,b]}$ minimizes the action among all the arcs having the same endpoints that are homotopic to $q_{|[a,b]}$.*



Remark. Observe that in the other sense, it can be proved for a Tonelli Hamiltonian the following result (see [2]).
if all the projections of pieces of orbits are minimizing among the arcs in the same homotopy class with fixed endpoints, then the space is foliated into invariant Lipschitz Lagrangian graphs (see section 5 for a definition).

We can ask ourselve if the class of the invariant Lagrangian graphs is not a very small subclass of the one of Lagrangian submanifolds. But in fact :

Theorem. *(M.-C. Arnaud, [4]) Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Tonelli Hamiltonian and let $\mathcal{N}$ be a Lagrangian submanifold that is invariant by the Hamiltonian flow of H. If $\mathcal{N}$ is* Hamiltonianly isotopic *to a graph, then $\mathcal{N}$ is a graph.*

Remark. There exist some Tonelli Hamiltonians that have some invariant submanifolds that are

- homotopic to $\mathbb{T}^n \times \{0\}$;
- not Lagrangian;
- not graphs.

Hence we need the hypothesis "Lagrangian submanifold" in the previous theorem. The following example is given in [7].
Let $H : \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the Hamiltonian defined by: $H(\theta, r) = \frac{1}{2}(r_1^2 + r_2^2 + r_3^2)$ and let $j = (j_1, j_2) : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$ be a smooth embedding that is isotopic[3] to $\theta \to (\theta, 0)$

[3] Two embeddings are isotopic if they are homotopic in the set of the embeddings.

but is not a graph. Let $J : \mathbb{T}^3 \to \mathbb{T}^3 \times \mathbb{R}^3$ be the embedding map defined by:

$$J(\theta) = (\theta_1, \theta_2, j_1(\theta_3); j_2(\theta_3), 0, 0).$$

Then the submanifold $\mathcal{T} = J(\mathbb{T}^3)$, that is not a graph, is invariant by the Hamiltonian flow of H, non-Lagrangian and isotopic to $\mathbb{T}^3 \times \{0\}$.

5. Minimization and phantom of invariant Lagrangian graphs

For a general Tonelli Hamiltonien system, there is no foliation of the manifold into invariant Lagrangian graphs. Contreras, Figalli and Rifford gave an example of a Riemannian metric on $\mathbb{T}^2$ that has only one invariant Lagrangian graph (see [12]).

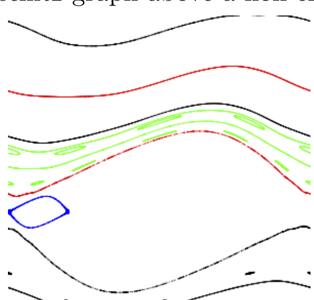
Looking for the phantoms of these invariant tori, John Mather (see [16]) had the idea to consider this problem from the Lagrangian side. Then he tried to find action minimizing orbits or measures. We defined in the previous section what is the Lagrangian action of a curve, let us define what is the action of a Borel probability measure $\mu = \mathcal{L}^*\nu$ where ν is a Borel probability that is invariant by the Hamiltonian flow,

$$\mathcal{F}(\mu) = \int_{\mathbb{T}^n \times \mathbb{R}^n} L(q,v)d\mu(q,v).$$

To any invariant Borel probability with bounded support, we can associate its *rotation vector*

$$\rho(\mu) = \int_{\mathbb{T}^n \times \mathbb{R}^n} v d\mu(q,v).$$

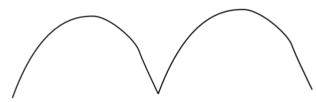
J. Mather minimizes the action among the invariant measures that have a fixed rotation number. Then he proves that the union of the supports of all the minimizing measures with fixed rotation number ρ, which is called the Mather set with rotation number ρ, is a Lipschitz graph above a non-empty subset of $\mathbb{T}^n$.



In dimension 2, the analogues of the Mather sets in the discrete case are Lipschitz graphs above: finite sets, Cantor set or the whole circle

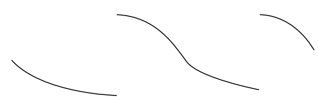
But these Mather sets can be very small (sometimes one point). Albert Fathi (see [13]) proved that they can be put in some larger sets that are in some weak sense Lagrangian graphs,

- a Mather set can always be embedded in a Lipschitz Lagrangian graph that is in general not invariant by the Hamiltonian flow (this result is due to P. Bernard, see [10] and also [5]): a Lipschitz function being almost everywhere differentiable (this is Rademacher Theorem), we can define at almost every point of a Lipschitz graph its tangent subspace. We ask that this tangent subspace is Lagrangian; such a graph is the graph of $a + du$ where $a \in \mathbb{R}^n$ and $u \in C^{1,1}(\mathbb{T}^n, \mathbb{R})$;



A Lipschitz Lagrangian graph

- a Mather set can always be embedded in a discontinuous Lagrangian graph $\mathcal{N}$ that is positively invariant by the Hamiltonian flow: $\forall t > 0, \varphi_t^H(\mathcal{N}) \subset \mathcal{N}$. More precisely, such a graph is called a pseudograph and is the graph of $a + du$ where $u : \mathbb{T}^n \to \mathbb{R}$ is semi-convex, i.e. in charts the sum of a convex function and a C^2 one.



A 1-dimensional pseudograph: as the derivative of a convex function is increasing, the jumps at every discontinuity are positive.

Such a pseudograph has then a partition into half orbits $\varphi_{]0,\infty[}(x)$.

Let us mention the following point for dynamicists: if for example the Mather set is a hyperbolic equilibrium, then the corresponding pseudograph is a part of its stable manifold.

6. Shape and dynamics

Here we will show that there is a link between the dynamics and the regularity of the support of a minimizing measure. We have seen that certain dynamics force a graph to be Lagrangian. Now we will explain that certain dynamics force a Lagrangian graph to be regular.

Lyapunov exponents. Let μ be an invariant Borel probability measure for the Hamiltonian C^1 flow (φ_t). Then there exists $\lambda_k > \cdots > \lambda_1 \geq 0 \geq -\lambda_1 > \cdots > -\lambda_k$ $2k$ real numbers that are called *Lyapunov exponents* and at μ-almost everypoint a measurable splitting invariant by $D\varphi_t^H$

$$T_x(\mathbb{T}^n \times \mathbb{R}^n) = E_k^u(x) \oplus \cdots \oplus E_1^u(x) \oplus E^c(x) \oplus E_1^s(x) \oplus \cdots \oplus E_k^s(x)$$

such that

$$\forall v \in E_i^u(x)\backslash\{0\}, \lim_{t \to \pm\infty} \frac{1}{t} \log\left(\|D\varphi_t^H(x).v\|\right) = \lambda_i;$$

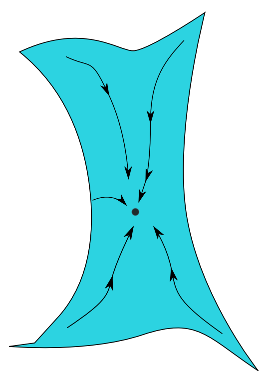
$$\forall v \in E_i^s(x)\backslash\{0\}, \lim_{t \to \pm\infty} \frac{1}{t} \log\left(\|D\varphi_t^H(x).v\|\right) = -\lambda_i;$$

and

$$\forall v \in E^c(x)\backslash\{0\}, \lim_{t\to\pm\infty} \frac{1}{t} \log\left(\|D\varphi_t^H(x).v\|\right) = 0$$

These Lyapunov exponents measure the exponential rate of dilatation or contraction of vectors. To the non-zero Lyapunov exponents there corresponds *stable or unstable sumanifolds* if H is at least C^3, i.e. exist for μ-almost every point x

- an (immersed) stable submanifold $W^s(x)$ that is tangent to $E^s(x) = E_1^s(x) \oplus \dots \oplus E_k^s(x)$ and such that
$$\forall y \in W^s(x), \lim_{t\to+\infty} d(\varphi_t^H(x), \varphi_t^H(y)) = 0;$$
- an (immersed) unstable submanifold $W^u(x)$ that is tangent to $E^u(x) = E_1^u(x) \oplus \dots \oplus E_u^s(x)$ such that
$$\forall y \in W^u(x), \lim_{t\to-\infty} d(\varphi_t^H(x), \varphi_t^H(y)) = 0.$$



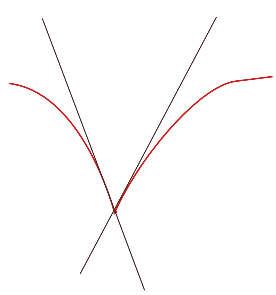
The stable submanifold of a fixed point

Let us now introduce a notion that extends the one of tangent space to a submanifold to any subset.

When $A \subset \mathbb{R}^{2n}$ and $a \in A$, the *contingent cone* $\mathcal{C}_a A$ of A at a is the set of the limits

$$\lim_{k\to\infty} \frac{1}{\lambda_k}(a_k - a) = v.$$

where, for every $k \in \mathbb{N}$, $\lambda_k \in \mathbb{R}$ and $a_k \in A \to a$.
Using charts, we can extend this definition to subsets of any manifolds, as $\mathbb{T}^n \times \mathbb{R}^n$.



A contingent cone

- We can also define the *limit contingent cone* $\mathcal{L}_aA$ that is in some sense the limsup of all the contingent cones $\mathcal{C}_bA$s for $b \in A$ tending to a.
- A subset $A \subset \mathbb{T}^n \times \mathbb{R}^n$ is C^1 at $a \in A$ if its limit contingent cone $\mathcal{L}_aA$ is contained in some Lagrangian subspace.

Hence A is C^1 if its tangent space is not too large (and is Lagrangian). Observe that in $\mathbb{T}^n \times \mathbb{R}^n$, the Lipschitz Lagrangian graph is C^1 if and only if it is the graph of a C^1 function (in the usual sense).

Theorem. *(M.-C. Arnaud, [6]) If all the Lyapunov exponents of a minimizing measure for a Tonelli Lagrangian are zero, then the support of this minimizing measure is the graph of map that is C^1 μ-almost everywhere.*

Hence, if the support of a minimizing measure is irregular (in the C^1-sense) μ-almost everywhere, the measure has at least one non-zero exponent (and thus non-trivial stable and unstable submanifolds).

The shape of the support of μ is related to some dynamical characteristics. There are some variations of this result (see [3]):

If a Lipschitz Lagrangian $\mathcal{G}$ graph is invariant and the dynamics on $\mathcal{G}$ is equi-Lipchitz (i.e. the $(\varphi^H_{t|\mathcal{G}})$ are equiLipschitz), then the graph is C^1.

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