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Green bundles and related topics

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Abstract. For twist maps of the annulus and Tonelli Hamiltonians, two linear bundles, the Green bundles, are defined along the minimizing orbits.
The link between these Green bundles and different notions as : weak and strong hyperbolicity, estimate of the non-zero Lyapunov exponents, tangent cones to minimizing subsets, is explained.
Various results are deduced from these links : the relationship between the hyperbolicity of the Aubry-Mather sets of the twist maps and the C^1-regularity of their support, the almost everywhere C^1-regularity of the essential invariant curves of the twist maps, the link between the Lyapunov exponents and the angles of the Oseledec bundles of minimizing measures, the fact that C^0-integrability implies C^1-integrability on a dense G_δ-subset.

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1. Introduction

In the study of twist maps or optical Hamiltonians, mathematicians have studied the orbits that can be found via minimization for a long time : an action is associated with such a dynamical system, and an orbit piece corresponds to a critical point of the action. For example, a way to find periodic orbits is to minimize the action among the periodic arcs (for Hamiltonians) or sequences (for twist maps).

More recently, the existence of some globally minimizing orbits has been proved, i.e. the existence of orbits that minimize the action along all the intervals of time. In the case of twist maps, these orbits are contained in some minimizing sets (i.e. sets filled with minimizing orbits) called Aubry-Mather sets, which were independently discovered in the 80's by S. Aubry & P. Le Daeron and J. Mather. In the case of the so-called Tonelli Hamiltonians, their existence was proved by J. Mather in the 90's when he proved the existence of minimizing measures. In the case of a Tonelli Hamiltonian of a cotangent bundle T^*M, some minimizing sets similar to the Aubry-Mather sets, called Aubry sets, also exist.

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To have an idea of what these Aubry-Mather sets may be, let us consider the case of a completely integrable twist map of $\mathbb{A} = \mathbb{T} \times \mathbb{R} : f \ : (q, p) \to (q + d\tau(p), p)$. Then the annulus is foliated by invariant circles $\{p = C\}$, which are the Aubry-Mather sets. If we slightly perturb f, a lot of these invariant curves will persist (this is a consequence of the K.A.M. theorems), but some others will become smaller invariant sets, Cantor sets or periodic orbits; these three kinds of sets are Aubry-Mather sets; in a certain way, they are the ghosts of the initial invariant circles.

In the case of a generic twist map of the annulus, a result due to Patrice Le Calvez states that the majority of Aubry-Mather sets are hyperbolic (see [30]). No such result is known for the Tonelli Hamiltonians.
In the case of twist maps, too, we know that some non-hyperbolic Aubry-Mather sets, the K.A.M. curves, may persist after perturbation.

We can then ask ourselves :

Question 1. *is there a means of distinguishing between the hyperbolic and the non hyperbolic Aubry or Aubry-Mather sets? Is there a means of* seeing *the Lyapunov exponents of a minimizing measure when knowing only the measure and not the dynamic?*

For the twist maps, there are three kinds of Aubry-Mather sets :
– the invariant curves, which are never uniformly hyperbolic;
– the periodic orbits, which may be hyperbolic or non hyperbolic; there is, of course, no way to distinguish between a hyperbolic and a non hyperbolic finite orbit if we only know the orbit;
- the Cantor sets, which may be hyperbolic or non hyperbolic; we will give a way to distinguish between hyperbolic and non hyperbolic Cantor sets.
Hence, in the case of twist maps, we obtain a criterion to decide if an Aubry-Mather set is hyperbolic or not, without knowing the dynamic. To be a little more precise, we define a notion of C^1-regularity for the subsets of a manifold, and we prove that hyperbolicity is equivalent to C^1-irregularity. In the case of Tonelli Hamiltonians, we will see that this result is no longer true, but a partial result subsists.

The main tool to prove this kind of result is what is called the pair (G_+, G_-) of Green bundles that are defined along every minimizing orbit. These Lagrangian bundles were introduced by L. Green in 1958 in [26] for geodesic flows to prove some rigidity results. More precisely, the existence of only one Lagrangian invariant bundle transverse to the vertical (for example one Green bundle) is required in order to obtain some rigidity results in Riemannian geometry.

Then these Green bundles were used to characterize the Anosov geodesic flows (see [16], and [29] for related results). In this article, we will be interested in this kind of more dynamical result.

We will introduce the Green bundles, give their main properties and explain what kind of results were recently obtained through their use. Roughly speaking, the Green bundles are the limits of the successive images of the "verticals". Let us mention that in the 30's, G. Birkhoff already used the images of the verticals to obtain some a priori inequalities for the invariant curves of twist maps, i.e. to obtain some Lipschitz regularity results. We will speak of the relations between Green bundles and hyperbolicity, Green bundles and Lyapunov exponents and Green bundles and regularity.

The main definitive results that we give are :
• the characterization of the weak (strong) hyperbolicity of the minimizing measures of twist maps by the C^1-irregularity of their support (section 5); this gives a way to see hyperbolicity;
• for the minimizing measures, the link between the positive Lyapunov exponents and the angle between the Oseledec bundles (section 3); this relation is specific to the case of a twisting dynamical system and doesn't exist for general dynamical systems; this is a way to see the Lyapunov exponents;
• some regularity results for the invariant graphs that are C^0-Lagrangian when we make some dynamical assumptions (see section 4). Roughly speaking, we will see that a slow dynamic implies some regularity.

Let us give the outline of this article :
• in section 2, we recall some well-known facts concerning the twist maps and the Tonelli Hamiltonians, construct the Green bundles, and prove that uniform hyperbolicity is equivalent to the transversality of the Green bundles;
• in section 3, we characterize the number of zero Lyapunov exponents of a minimizing measure by way of the dimension of the intersection of the two Green bundles, and we give some estimates of the non-zero Lyapunov exponents via the angle between the Oseledec bundles;
• in section 4, we explain the relationship between the Green bundles and some cones that are tangent to the minimizing subsets. We deduce some regularity results such as : every continuous invariant graph of a twist map is C^1 almost everywhere; every C^0-integrable Tonelli Hamiltonian is C^1-integrable on an invariant dense G_δ-subset;
• in section 5 we give a complete explaination of the case of the Aubry-Mather sets of twist map : if they have no isolated point, their (weak or uniform) hyperbolicity is equivalent to their C^1-irregularity;
• in section 6, we give an overview of weak KAM theory;
• in section 7, we explain the link between the weak KAM solutions and the Green bundles. We deduce that the support of any minimizing measure all of whose Lyapunov exponents are zero is almost everywhere C^1-regular.

Let us mention that twist maps and Tonelli Hamiltonians appear in numerous problems issued from physics : motivated by the restricted 3-body problem, H. Poincaré introduced the twist maps at the end of the 19th century. Moreover, all the mechanical systems, that correspond to a Hamiltonian that is the sum of a kinetic energy and a potential energy are Tonelli Hamiltonian (N-body problems, simple pendulum. . .).

2. Green bundles and uniform hyperbolicity

In this section, we will define the two Green bundles along locally minimizing orbits and prove that their transversality implies some hyperbolicity.

2.1. Well-known facts for twist maps. A *twist map* of the annulus $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ is a C^1-diffeomorphism $f\ : (q,p) \to (Q,P)$ of $\mathbb{A}$ that is is isotopic to identity, satisfies the twist condition :
(twist) : for any lift $F\ : \mathbb{R}^2 \to \mathbb{R}^2$ of f, if we write : $F(x,y) = (X,Y)$, then the map : $(x,y) \to (x, X(x,y))$ is a C^1 diffeomorphism;
and has a global generating function $S\ : \mathbb{A} \to \mathbb{R}$ i.e. such that : $PdQ - pdq = dS$.
The inverse of a twist map is a twist map. The twist map f is *positive* if the map $(x,y) \to (x, X(x,y))$ is orientation preserving.
In general , the generating function is expressed in the lifted coordinates (x,X) : $s(x,X) = S \circ p(x,y)$ where $p\ : \mathbb{R}^2 \to \mathbb{A}$ is the covering map. We can associate the action functionals $A_{n,m}$ with this generating function :
given a sequence $(x_n, \dots x_m)$ of points of $\mathbb{R}$, its action is defined by :
$A_{n,m}(x_n, \dots, x_m) = \sum_{k=n}^{m-1} s(x_k, x_{k+1})$. Then $(x_n, \dots, x_m)$ is the projection of an orbit segment of F on the x-axis if, and only if, it is is a critical point of $A_{n,m}$ restricted to the space of sequences $(z_n, \dots, z_m)$ with fixed endpoints : $z_n = x_n$ and $z_m = x_m$. In this case the corresponding orbit segment is $(x_k, y_k)_{n \leq k \leq m}$ where $y_k = \frac{\partial s}{\partial X}(x_{k-1}, x_k) = -\frac{\partial s}{\partial x}(x_k, x_{k+1})$. A bi-infinite sequence $(x_k)_{k \in \mathbb{Z}}$ is (globally) minimizing if for any $[n,m] \subset \mathbb{Z}$, $(x_k)_{n \leq k \leq m}$ minimizes the action with fixed endpoints. In this case, it is the projection of a unique orbit of F, and we usually say that the corresponding orbit of f is minimizing. We say that $(x_k)_{k \in \mathbb{Z}}$ is locally minimizing if, for any $[n,m] \subset \mathbb{Z}$, $(x_k)_{n \leq k \leq m}$ locally minimizes the action with fixed endpoints.
In the 80's, J. Mather and S. Aubry & P. Le Daeron proved the existence of minimizing orbits (see [5], [34]). Moreover, they proved that every such minimizing orbit is contained in an invariant compact Lipschitz graph above a part of $\mathbb{T}$ that is the union of some minimizing orbits. These Lipschitz graphs are called Aubry-Mather sets. A very important property of these Aubry-Mather sets is that the projected dynamic (on $\mathbb{T}$) of the dynamic restricted to one of these Aubry-Mather set is the restriction of an orientation preserving bi-Lipschitz homeomorphism of the circle. Hence, we can associate a rotation number with such an Aubry-Mather set. We don't give a precise definition here because we won't need it, but the reader can find more details in [25].
Let us recall that an invariant probability μ is ergodic if the μ-measure of every invariant subset is 0 or 1. An ergodic Borel probability measure with compact support is said to be minimizing if its support contains only minimizing orbits. Then its support is an Aubry-Mather set.

2.2. Well-known facts for Hamiltonians. We may define Tonelli Hamiltonians on the cotangent bundle of any closed manifold, but to avoid some com-

plications in the choice of the coordinates (via a Riemannian connection), we will assume that the manifold is $\mathbb{T}^d$. Then $\mathbb{T}^d$ is endowed with its usual flat Riemannian metric and we denote by $\pi\ : (q,p) \in \mathbb{T}^d \times \mathbb{R}^d \to q \in \mathbb{T}^d$ the usual projection. A Tonelli Hamiltonian of $\mathbb{T}^d \times \mathbb{R}^d$ is a C^3 map $H\ : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ that is super-linear in the fiber and C^2 strictly convex in the fiber (i.e. the Hessian $H_{p,p}$ is positive definite at every point).
This Hamiltonian defines a Hamiltonian flow (φ_t) on $\mathbb{T}^d \times \mathbb{R}^d$, solution to the Hamilton equations :

$$\dot q = \frac{\partial H}{\partial p}(q,p); \quad \dot p = -\frac{\partial H}{\partial q}(q,p).$$

Let us point out a nice interpretation of the convexity assumption : if $V(q,p) = \ker D\pi(q,p) \subset T_{(q,p)}(\mathbb{T}^d \times \mathbb{R}^d)$ is the linear vertical, for all small enough t, the image $D\varphi_t V(q,p)$ of a vertical by the linearized flow is a Lagrangian subspace transverse to the vertical, a graph of a symmetric matrix, $s_t(\varphi_t(q,p))$, close to $\frac{1}{t}\frac{\partial^2 H}{\partial p^2}(q,p)$; moreover, as long as these images are transverse to the vertical, the family $(s_t(q,p))$ is decreasing for the natural order of the symmetric matrices.

We can associate its Legendre map $\mathcal{L}\ : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$, defined by : $\mathcal{L}(q,p) = (q, \frac{\partial H}{\partial p}(q,p))$ with such a Tonelli Hamiltonian. This Legendre map is a C^2-diffeomorphism. We can define too the Lagrangian $L\ : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ associated with H, defined by :

$$L(q,v) = \max_{p \in T_q^* M}(p.v - H(q,p)) = \mathcal{L}^{-1}(q,v).v - H \circ \mathcal{L}^{-1}(q,v).$$

The function L is then as regular as H is and $\gamma\ : I \to M$ is the projection of an orbit segment of the Hamiltonian flow of H if, and only if, $(\gamma,\dot\gamma)$ is a solution to the Euler-Lagrange equations associated with L :

$$\dot q = v; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial v}(q,v)\right) = \frac{\partial L}{\partial q}(q,v).$$

In this case, the corresponding orbit for the Euler-Lagrange flow (f_t) is given by : $t \to (\gamma(t),\dot\gamma(t))$ and the corresponding orbit for the Hamiltonian flow is : $t \to \mathcal{L}^{-1}(\gamma(t),\dot\gamma(t))$. Hence, the two flows are conjugated by $\mathcal{L}$.
An arc $\gamma\ : [a,b] \to \mathbb{T}^d$ gives a solution $(\gamma,\dot\gamma)$ to the Euler-Lagrange equations if and only if it is a critical point of the Lagrangian action : $A(\gamma) = \int_a^b L(\gamma,\dot\gamma)$ restricted to the set of C^1 arcs with fixed endpoints. We say that $\gamma\ : \mathbb{R} \to \mathbb{T}^d$ is a minimizer if for any segment $[a,b]$, $\gamma_{|[a,b]}$ minimizes the Lagrangian action among the C^1-curves having the same endpoints. We say that γ is a local minimizer if for any segment $[a,b]$, $\gamma_{|[a,b]}$ is a local (for the C^0-topology) minimizer of the action defined on the set of C^1-arcs with fixed endpoints. The corresponding orbits will be called (locally) minimizing orbits. J. Mather proved the existence of minimizing orbits in [35], and the existence of minimizing measures with compact support, which are ergodic invariant Borel probability measures whose support contains only minimizing orbits.

A classical result asserts that a curve $\gamma\ :[a,b]\to\mathbb{T}^d$ is locally minimizing if, and only if, γ is a solution to the Euler-Lagrange equations and $\mathcal{L}^{-1}(\gamma,\dot\gamma)_{|]a,b[}$ has no conjugate points; two points $x,y\in\mathbb{T}^d\times\mathbb{R}^d$ are conjugate if $t\neq 0$ exists so that $\varphi_t(x)=y$ and $D\varphi_t V(x)\cap V(y)\neq 0$.

A way to obtain a lot of locally minimizing measures and orbits is to use the so-called modified Lagrangians : if η is a closed 1-form of $\mathbb{T}^d$, then $L-\eta$ has the same Euler-Lagrange flow as L but not the same minimizing orbits. Hence, the two flows have the same locally minimizing orbits (because they have the same pairs of conjugate points) but not the same minimizing orbits. Using a lot of cohomologically different 1-forms, one finds a lot of locally minimizing orbits and measures.

2.3. Construction of the Green Bundles, first properties. There is a canonical way to identify $T_{(q,p)}(\mathbb{T}^d\times\mathbb{R}^d)$ with $\mathbb{R}^d\times\mathbb{R}^d$. In these coordinates, we have $V(q,p)=\{0\}\times\mathbb{R}^d$; a Lagrangian subspace is said to be *horizontal* if it is transverse to the vertical and $h(q,p)=\mathbb{R}^d\times\{0\}$ is such a horizontal subspace. Let us recall that the graph of a linear map $S\ :h(q,p)\to V(q,p)$ is Lagrangian if, and only if, S is symmetric.

In the tangent space along a locally minimizing orbit, we may define two invariant horizontal Lagrangian bundles, called the Green bundles. These bundles were introduced by L. Green in in 1958 in [26] for geodesic flows. P. Foulon extended the construction to the Finsler metrics in [21] and G. Contreras & R. Iturriaga built them for any Tonelli Hamiltonian in [14]. The construction for the twist maps of the annulus, and more generally for the twist maps of $\mathbb{T}^d\times\mathbb{R}^d$ is due to M. Bialy & R. MacKay (see [9])

The method is the following. We consider a locally minimizing orbit (x_t) where t is a real number or an integer, and at x_0 we construct the family of Lagrangian subspaces that are the images of the verticals by the linearized dynamical system : $G_t(x_0)=D\varphi_t V(x_{-t})$ where (φ_t) designates the Hamiltonian flow or the positive twist map. Because the orbit is minimizing, it has no conjugate points and then all the $G_t(x_0)$ for $t\neq 0$ are transverse to the vertical. There is a natural partial order for the Lagrangian subspaces that are transverse to the vertical. On the annulus, this order is the usual order between the slopes of the lines. In higher dimensions, it corresponds to the usual order for the symmetric operators associated with Lagrangian subspaces that we have just defined. For this relation, the family $(G_t)_{t>0}$ is a decreasing family of Lagrangian subspaces and $(G_{-t})_{t>0}$ is an increasing family. Moreover, we have : $\forall u,t>0, G_{-u}(x_0)\leq G_t(x_0)$. We have then an increasing sequence of Lagrangian subspaces bounded from above and a decreasing one bounded from below. We can take the limit and define the two Green bundles :

$$G_-(x_0)=\lim_{t\to+\infty}G_{-t}(x_0);\quad G_+(x_0)=\lim_{t\to+\infty}G_t(x_0).$$

These two bundles are Lagrangian, invariant, transverse to the vertical and satisfy :

$$\forall t>0, G_{-t}(x_0)<G_-(x_0)\leq G_+(x_0)<G_t(x_0).$$

Being the limits of monotone sequences of continuous bundles, these bundles are semicontinuous (see [1], [3]) : G_- is lower semicontinuous and G_+ is upper semicontinuous.
Remark. in the case of a Tonelli Hamiltonian, an orbit has no conjugate points if, and only if, there exists an invariant Lagrangian sub-bundle F along this orbit that is transverse to the vertical. In this case, we have along the orbit : $G_- \leq F \leq G_+$.

2.4. A dynamical criterion and some consequences. The orbit (x_t) being locally minimizing and relatively compact, there is a classical result that gives a way to prove that some vectors are in one of the two Green bundles :

Proposition 1 (dynamical criterion). *Let $v \in T_{x_0}(\mathbb{T}^d \times \mathbb{R}^d)$. Then :*
– if $v \notin G_-(x_0)$, then $\lim_{t\to+\infty} \|D\pi \circ D\varphi_t.v\| = +\infty$;
–if $v \notin G_+(x_0)$, then $\lim_{t\to+\infty} \|D\pi \circ D\varphi_{-t}.v\| = +\infty$.

Proof. To prove the first point of the proposition, we express the matrix of $D\varphi_t$ in the global coordinates of $\mathbb{R}^d \times \mathbb{R}^d$. Then, we use a linear symplectic change of coordinates along the orbit so that the "horizontal subspace" becomes G_-. Because G_- is between G_{-1} and G_1, which depend continuously on x, this change of coordinates is bounded. Then in these coordinates the matrix of $D\varphi_t(x_0)$ is :

$$M_t(x_0) = \begin{pmatrix} -b_t(x_0)s_{-t}(x_0) & b_t(x_0) \\ 0 & s_t(x_t)b_t(x_0) \end{pmatrix}$$

where $G_t(x)$, which is Lagrangian, is the graph of the symmetric matrix $s_t(x)$. The matrix being symplectic, we have : ${}^tb_t(x_0)s_t(x_t)b_t(x_0) = -(s_{-t}(x_0))^{-1}$. As $(s_t(x_t))_{t\geq 1}$ is bounded by s_{-1} and s_1 and as $(s_{-t}(x_0))_{t>0}$ tends to 0 from below when t tends to $+\infty$, we deduce that $\lim_{t\to\infty} m(b_t) = +\infty$ where $m(b) = \|b^{-1}\|^{-1}$ designates the conorm of b. From this we deduce immediately that if a vector $v = (v_1, v_2)$ is not in G_-, i.e. if $v_2 \neq 0$, then : $\lim_{t\to+\infty} \|D\pi \circ D\varphi_t.v\| = +\infty$. □

Remark. 1) We deduce from the dynamical criterion that in the Hamiltonian case, the Hamiltonian vector-field X_H belongs to the two Green bundles. Because these two Green bundles are Lagrangian, this implies that G_+ and G_- are tangent to the Hamiltonian levels $\{H = c\}$.
2) Moreover, we deduce, too, that if there is an Oseledec splitting (this will be defined precisely in section 3), $T(\mathbb{T}^d \times \mathbb{R}^d) = E^s \oplus E^c \oplus E^u$ above a invariant compact set K without conjugate points, then $E^s \subset G_-$ and $E^u \subset G_+$. Because the flow is symplectic, E^u and E^s are isotropic and orthogonal to E^c for the symplectic form (see [10]). We deduce that $G_- \subset E^s \oplus E^c$, $G_+ \subset E^u \oplus E^c$ and then $G_- \cap G_+ \subset E^c$. Hence, $G_- \cap G_+$, being an isotropic subspace of the symplectic subspace E^c, we obtain : $\dim E^c \geq 2\dim(G_- \cap G_+)$. The dimension of the intersection of the two Green bundles gives a lower bound of the number of zero Lyapunov exponents. We will soon prove that this inequality is, in fact, an equality.
We have the same results for a hyperbolic or partially hyperbolic dynamic. Let us

notice that in the hyperbolic case, G_- (resp. G_+) is nothing else but the stable (resp. unstable) bundle E^s (resp. E^u)
3) Let us consider the case of a K.A.M. torus that is a graph : the dynamic on this torus is C^1 conjugated to a flow of irrational translations on the torus $\mathbb{T}^d$; M. Herman proved in [28] that such a torus is Lagrangian, and it is well-known that any invariant Lagrangian graph is locally minimizing. Then the orbit of every vector tangent to the K.A.M. torus is bounded, and belongs to $G_- \cap G_+$. In this case, the two Green bundles are equal to the tangent space of the invariant torus.

The dynamical criterion is the key argument for proofs of hyperbolicity results. In [16] , [21], [14], the authors prove that if there is no conjugate points in a whole energy level and if the Green bundles are transverse in the tangent space of the energy level, then the flow restricted to this energy level is Anosov. In fact, we may extend these results to the locally minimizing subsets. Let us recall :
a subset $K \subset \mathbb{A}$ that is invariant by a twist map f is *hyperbolic* if along K there is an Df-invariant splitting $T_x\mathbb{A} = E^s \oplus E^u$ so that along the stable bundle E^s, Df is uniformly contracting and along the unstable bundle E^u, Df is uniformly expanding. A subset $K \subset \mathbb{T}^d \times \mathbb{R}^d$ is *partially hyperbolic* for the Tonelli flow (φ_t) if there is an invariant splitting $E^s \oplus E^c \oplus E^u$ such that $D\varphi_{t|E^s}$ is uniformly contracting, $D\varphi_{t|E^u}$ is uniformly expanding and $D\varphi_{t|E^c}$ is less contracting than $D\varphi_{t|E^s}$ and less expanding than $D\varphi_{t|E^u}$.
For an Hamiltonian flow, as the flow direction and the energy direction are in E^c, we always have : $\dim E^c \geq 2$.

Theorem 2 (Green bundles and uniform hyperbolicity). *Let K be a compact invariant locally minimizing set. Then :*
• *in the case of a twist map, K is hyperbolic if, and only if, at all points of K, G_+ and G_- are transverse;*
• *in the case of a Hamiltonian flow, if K contains no singularity, K is partially hyperbolic with a center bundle with dimension 2 if, and only if, at all points of K, G_- and G_+ are transverse in the energy level.*

Proof. Let us outline the ideas of the proof in the direct sense in the case of transversality of the Green bundles (for example for twist maps). In the Hamiltonian case, where these two bundles are not transverse in the whole tangent space, we restrict and reduce the dynamic to obtain a symplectic cocycle on $T\mathcal{E}/\mathbb{R}X_H$ where $\mathcal{E}$ designates the energy level : for this symplectic reduced cocycle, there exist two reduced Green bundles that are transverse (see [2]). Hence, we only have to prove the result for transverse Green bundles and symplectic cocycles.
In this case, the dynamical criterion implies that the cocycle is *quasi-hyperbolic*, i.e. that the orbit of any non null vector under the cocycle is unbounded. Quasi-hyperbolic dynamics (or more precisely quasi-Anosov dynamics) were studied by R. Mañé in [31]. Quasi-Anosov dynamics that are not Anosov exist (see [22], [38]), but we proved in [3] that a quasi-hyperbolic symplectic cocycle is hyperbolic. The proof mainly uses the original ideas of Mañé. □

Remark. We have seen that in the hyperbolic case, the Green bundles G_-, G_+

are equal to the stable/unstable bundles E^s, E^u. We have seen, too, that along a KAM curve, the two Green bundles are equal. We deduce from these remarks and from the fact that the two Green bundles are semicontinuous that : if T is a KAM curve and if $\varepsilon > 0$ is a positive number, a neighborhood U of T exists so that along any hyperbolic invariant locally minimizing set K contained in U, the two Oseledec bundles E^s and E^u are ε-close to each other. In section 3, we will give a refinement of these remarks by giving some inequalities between the Lyapunov exponents and the angle between the stable and unstable Oseledec bundles.

3. Non-uniform hyperbolicity, estimations of the Lyapunov exponents of minimizing measures

In this section, we will speak of the link between the angle of the two Green bundles and the Lyapunov exponents of a locally minimizing measure.

If K is an invariant subset of a Tonelli flow or twist map denoted by (φ_t), we will say that there is an Oseledec splitting on K if there exist $\lambda_1 < \cdots < \lambda_m$ and an invariant splitting $T(\mathbb{T}^d \times \mathbb{R}^d) = E_1 \oplus \cdots \oplus E_m$ with constant dimensions above K so that :

$$\forall x \in K, \forall i \in [1, m], \forall v \in E_i(x), \lim_{t\to+\infty} \frac{1}{t} \log \|D\varphi_t v\| = \lim_{t\to-\infty} \frac{1}{t} \log \|D\varphi_t v\| = \lambda_i.$$

The Lyapunov exponents are then the λ_i. The stable bundle is $E^s = \bigoplus_{\lambda_i<0} E_i$, the center bundle is $E^c = \bigoplus_{\lambda_i=0} E_i$ and the unstable bundle is $E^u = \bigoplus_{\lambda_i>0} E_i$. The integer $\dim E_i$ is the multiplicity of λ_i. In the symplectic case, 0 always has an even multiplicity, and the Lyapunov exponents λ and $-\lambda$ have the same multiplicity. In the symplectic case, E^s and E^u are isotropic for the symplectic form and E^c is symplectic and orthogonal to $E^s \oplus E^u$ (see [10]).

Oseledec's theorem ([37]) asserts : if μ is an invariant ergodic Borel probability with compact support of (φ_t), then there exists an invariant subset K with full μ-measure so that there exists an Oseledec splitting on K. The corresponding Lyapunov exponents are called the Lyapunov exponents of μ.

In the case of a discrete dynamical system, we say that the measure μ is *weakly hyperbolic* if all its Lyapunov exponents are non zero. If (φ_t) is a symplectic flow, then the multiplicity of the exponent zero is at least 2 (in the directions of the flow and of the energy), and we say that the measure is *weakly hyperbolic* if $\dim E^c = 2$. In this case, the extended stable and unstable bundles are $\tilde{E}^s = E^s \oplus \mathbb{R}X_H$ and $\tilde{E}^u = E^u \oplus \mathbb{R}X_H$ where X_H designates the vector-field.

3.1. The link between the Green bundles and the number of zero Lyapunov exponents.

Let us now consider a minimizing measure. Because we have assumed that we are only looking at ergodic measures, we can associate its Lyapunov exponents with such a measure. Because the dynamic is symplectic, the number of positive exponents is equal to the number of negative exponents, and the number of zero exponents is even. We obtain in [3] and [2] the following result :

Theorem 3. *Let μ be a minimizing measure. Then μ has exactly 2ρ zero Lyapunov exponents if, and only if, at μ-almost every point, we have :* $\dim(G_-\cap G_+)=\rho$.

Proof. We assume that μ is a minimizing measure. The map $(q,p)\to\dim(G_-(q,p)\cap G_+(q,p))$ being measurable, invariant and μ being ergodic, it is constant almost everywhere. We denote its value by ρ. Let us assume for a while that $\rho=0$. We use then some coordinates analogous to the ones used in the proof of proposition 1, but our "horizontal bundle" is now G_+. Let us recall that the matrix of the linearized dynamic in these coordinates is :

$$M_t(x_0)=\begin{pmatrix}-b_t(x_0)s_{-t}(x_0) & b_t(x_0)\\ 0 & s_t(x_t)b_t(x_0)\end{pmatrix}$$

We denote by $s_\pm(x)$ the symmetric matrix whose Green bundle $G_\pm$ is the graph. Then $s_+=0$ and the matrix s_- is negative definite almost everywhere.
If the time is continuous, i.e. if the dynamical system is a Hamiltonian flow (φ_t), there exists a time $\tau>0$ such that μ is ergodic for the map φ_τ. We may assume that $\tau=1$ and from now we work on with the discrete dynamical system $(\varphi_k)_{k\in\mathbb{Z}}$. Using an Egorov theorem, we find for every $\varepsilon>0$ a measurable subset J_ε of $\mathbb{T}^d\times\mathbb{R}^d$ and two constants $\beta>\alpha>0$ so that $\mu(J_\varepsilon)>1-\varepsilon$, $(m(b_n))$ tends uniformly to $+\infty$ on J_ε, and : $\forall x\in J_\varepsilon, \beta\mathbf{1}\geq -s_-(x)\geq\alpha\mathbf{1}$ where $\mathbf{1}$ designates the (symmetric) matrix of identity. Then we choose $N\geq 0$ such that $\forall x\in J_\varepsilon, \forall n\geq N, m(b_n(x))\geq\frac{2}{\alpha}$.
Using the Birkhoff ergodic theorem, we know that for long intervals of time, the orbit piece of almost every point of J_ε comes back into J_ε in a proportion of time bigger than $1-2\varepsilon$. We deduce that there is, in such an orbit piece, a proportion bigger than $\frac{1-2\varepsilon}{N}$ of points that belong to J_ε and that correspond to times whose difference is more than N. If $x_0\in J_\varepsilon$ is a generic point for μ and if we denote by $m_1<m_2<\dots<m_n<\dots$ the return times in J_ε so that $m_{n+1}-m_n\geq N$, because the term $-b_n(x_0)s_{-n}(x_0)$ is multiplicative, for a big enough n, we find that $m(-b_{m_n}s_{m_n}(x_0))$ is greater than the product of $n\geq\frac{1-2\varepsilon}{N}.m_n$ terms greater than $\alpha.\frac{2}{\alpha}=2$. Hence, for almost every point of J_ε and for a big enough n, we obtain : $m(-b_{m_n}s_{m_n}(x_0))\geq\left(2^{\frac{1-2\varepsilon}{N}}\right)^{m_n}$. This implies that μ has at least d Lyapunov exponents greater than $\frac{1-2\varepsilon}{N}\log 2>0$ and finishes the proof in this case.
If $\rho\neq 0$, we have seen that μ has at least 2ρ zero exponents. Then, we consider the restricted-reduced cocycle on $(G_++G_-)/G_-\cap G_+$ and we prove that it is a symplectic cocycle whose Green bundles are transverse μ almost everywhere. We apply the previous result to find $2(d-\rho)$ positive Lyapunov exponents. □

3.2. Lower and upper bounds for the positive Lyapunov exponents in the Hamiltonian case. In the case of ergodic measures of a geodesic flow with support filled by locally minimizing orbits, i.e. in the case of measures with no conjugate points, A. Freire and R. Mané proved in [23] a nice formula for the sum of the positive exponents (see [21] and[14] too). A slight improvement of this formula gives :

Theorem 4. *Let μ be a Borel probability measure with no conjugate points that is ergodic for the Hamiltonian flow. If G_+ is the graph of $\mathbb{U}$ and G_- the graph of $\mathbb{S}$, the sum of the positive Lyapunov exponents of μ is equal to :*

$$\Lambda_+(\mu) = \frac{1}{2}\int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}(\mathbb{U} - \mathbb{S}))d\mu.$$

Hence, we see that the more distant the Green bundles, the greater is the sum of the positive Lyapunov exponents. This gives an upper bound to the positive Lyapunov exponents. A similar statement was given in the (non published) thesis of G. Kniepper.

We can be more precise by introducing a notion of symplectic angle :

Definition. Let F, G be two Lagrangian subspaces of a symplectic linear space (E, ω) endowed with an adapted scalar product . The *greatest angle* between F and G is defined by :

$$\beta(F, G) = \max_{(v,w)\in(F\backslash\{0\})\times(G\backslash\{0\})} \frac{\omega(v, w)}{\|v\|.\|w\|}.$$

This angle is equal to 0 if, and only if, $F = G$. Otherwise, it is positive. In particular, for a weakly hyperbolic measure, if $\tilde{E}^s = E^s \oplus \mathbb{R}X_H$ and $\tilde{E}^u = E^u \oplus \mathbb{R}X_H$ are the enlarged stable and unstable bundles of the Oseledec splitting, their greatest angle is positive.

To obtain some precise estimates of $\Lambda_+(\mu)$ by using this angle, we need some notations :

Notations. If $C > 0$ is a real number and $K \subset \mathbb{T}^d \times \mathbb{R}^d$ is a compact subset, we denote by $\mathcal{H}_C(K)$ the set of Tonelli Hamiltonians such that :

$$\forall x \in K, \exists t, u \in]0, 1], s_t(x) \leq C\mathbf{1} \quad \text{and} \quad s_{-u}(x) \geq -C\mathbf{1};$$

where s_t is the matrix of G_t and $\mathbf{1}$ denotes the matrix of identity.

Hence, we say that the elements of $\mathcal{H}_C(K)$ have a minimal twist of the vertical in K.

An easy consequence of theorem 4 is a formulation using the greatest angle :

Corollary 5. *Let $H \in \mathcal{H}_C(K)$ be a Tonelli Hamiltonian and let μ be a weakly hyperbolic ergodic Borel probability measure whose support in contained in K and has no conjugate points. Then :*

$$\frac{1}{2}\int m(\frac{\partial^2 H}{\partial p^2})\beta(\tilde{E}^s, \tilde{E}^u)d\mu \leq \Lambda_+(\mu) \leq \frac{d(1 + C^2)}{2}\int \mathrm{Tr}(\frac{\partial^2 H}{\partial p^2}).\beta(\tilde{E}^s, \tilde{E}^u)d\mu$$

where $\mathrm{Tr}(b)$ *designates the trace of b and $m(b) = \|b^{-1}\|^{-1}$ the conorm of b.*

Proof. A consequence of the linearized Hamilton equations is that if the graph $\mathcal{G}$ of a symmetric matrix G is invariant by the linearized flow, then any infinitesimal orbit $(\delta q, G\delta q)$ satisfies the following equation : $\delta\dot{q} = (\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q \partial p})\delta q$.
Hence, we have : $\frac{d}{dt}\det(D\pi \circ D\varphi_{t|\mathcal{G}}) = \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q\partial p})\det(D\pi \circ D\varphi_{t|\mathcal{G}})$; we deduce :
$\frac{1}{T}\log\det(D\pi \circ D\varphi_{T|\mathcal{G}}(q,p))$

$$= \frac{1}{T}\log\det(D\pi(q,p)_{|\mathcal{G}}) + \frac{1}{T}\int_0^T \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q\partial p})(\varphi_t(q,p))dt.$$

Via ergodic Birkhoff's theorem, we deduce for (q,p) generic that :

$$\liminf_{T\to+\infty}\frac{1}{T}\log\det(D\pi \circ D\varphi_{T|\mathcal{G}}(q,p)) = \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}G + \frac{\partial^2 H}{\partial q\partial p})d\mu.$$

Moreover, we know that $E^s \subset G_- \subset E^{s\perp} = E^c \oplus E^s$ and that $E^u \subset G_+ \subset E^{u\perp} = E^c \oplus E^u$. Hence, the sum of the Lyapunov exponents of the restricted cocycle $(D\varphi_{t|G_+})$ is exactly $\Lambda_+(\mu)$ and the sum of the Lyapunov exponents of the restricted cocycle $(D\varphi_{t|G_-})$ is $\Lambda_-(\mu) = -\Lambda_+(\mu)$. Then we have :

$$\Lambda_+(\mu) = \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}\mathbb{U} + \frac{\partial^2 H}{\partial q\partial p})d\mu \quad \text{and} \quad -\Lambda_+(\mu) = \int \mathrm{tr}(\frac{\partial^2 H}{\partial p^2}\mathbb{S} + \frac{\partial^2 H}{\partial q\partial p})d\mu$$

We obtain the conclusion by subtracting the two equalities. □

We obtain, too, a lower bound for the smallest positive Lyapunov exponent. To explain this, we need a notation :

Notations. If S is a positive semidefinite matrix, then $q_+(S)$ is its smallest positive eigenvalue.

Theorem 6. *Let μ be a measure with no conjugate points and with at least one non zero Lyapunov exponent; then its smallest positive Lyapunov exponent $\lambda(\mu)$ satisfies : $\lambda(\mu) \geq \frac{1}{2}\int m(\frac{\partial^2 H}{\partial p^2}).q_+(\mathbb{U}-\mathbb{S})d\mu$.*

Hence, the gap between the two Green bundles gives a lower bound of the smallest positive Lyapunov exponent. It is not surprising that when E^s and E^u collapse, the Lyapunov exponents are 0. What is more surprising and specific to the case of Tonelli Hamiltonians is the fact that the bigger the gap between E^s and E^u is , the greater the Lyapunov exponents are : in general, along a hyperbolic orbit, you may have a big angle between the Oseledec bundles and some very small Lyapunov exponents.

Proof. Let μ be an ergodic Borel probability measure with no conjugate points; its support K is compact and H belongs to some $\mathcal{H}_C(K)$. We choose a point (q,p) that is generic for μ and $(\delta q, \mathbb{U}\delta q)$ in the Oseledec bundle corresponding to the smallest positive Lyapunov exponent $\lambda(\mu)$ of μ. Using the linearized Hamilton equations, we obtain :

$$\frac{d}{dt}((\delta q(\mathbb{U}-\mathbb{S})\delta q) = \delta q(\mathbb{U}-\mathbb{S})\frac{\partial^2 H}{\partial p^2}(q_t,p_t)(\mathbb{U}-\mathbb{S})\delta q.$$

Hence :

$$\frac{d}{dt}((\delta q(\mathbb{U}-\mathbb{S})\delta q) \geq m(\frac{\partial^2 H}{\partial p^2})q_+(\mathbb{U}-\mathbb{S})\delta q(\mathbb{U}-\mathbb{S})\delta q;$$

and :
$\frac{2}{T}\log(\|\delta q(T)\|) + \frac{\log 2C}{T} \geq \frac{1}{T}\log(\delta q(T)(\mathbb{U}-\mathbb{S})(q_T,p_T)\delta q(T)) \geq$

$$\frac{1}{T}\log(\delta q(0)(\mathbb{U}-\mathbb{S})(q,p)\delta q(0)) + \frac{1}{T}\int_0^T m(\frac{\partial^2 H}{\partial p^2}(q_t,p_t))q_+((\mathbb{U}-\mathbb{S})(q_t,p_t))dt.$$

Using Birkhoff's ergodic theorem, we obtain :

$$\lambda(\mu) \geq \frac{1}{2}\int m(\frac{\partial^2 H}{\partial p^2})q_+(\mathbb{U}-\mathbb{S})d\mu.$$

□

3.3. The non negative Lyapunov exponent for twist maps.

In this case, we are interested in exactly one Lyapunov exponent. Hence, a formula giving the sum of the positive Lyapunov exponents is enough to bound the unique non negative Lyapunov exponent from below and above. Using the standard coordinates of $\mathbb{A}$, we obtain :

Theorem 7. *Let $f : \mathbb{A} \to \mathbb{A}$ be a positive twist map and let μ be a minimizing measure whose non negative Lyapunov exponent is λ. If s_-, s_+ designate the slopes of the two Green bundles and s_k designates the slope of G_k, we have :*

$$\lambda = \frac{1}{2}\int \log(\frac{s_+ - s_{-1}}{s_- - s_{-1}})d\mu = \frac{1}{2}\int \log(1 + \frac{s_+ - s_-}{s_- - s_{-1}})d\mu.$$

As in the Hamiltonian case, we see that the greatest/smallest exponent depends only on the deviation of the vertical (s_{-1} in the discrete case, $\frac{\partial^2 H}{\partial p^2}$ and C in the Hamiltonian case) and on the "angle" between the two Green bundles.

Proof. As in the proof of proposition 1, we use coordinates such that G_- is the horizontal bundle; the matrix of Df^n at x is then :

$$M = \begin{pmatrix} b_n(x)(s_-(x) - s_{-n}(x)) & b_n(x) \\ 0 & b_n(x)(s_n(f^n x) - s_-(f^n x)) \end{pmatrix}$$

We know that $G_- \subset E^s \oplus E^c$, hence along G_- we see the Lyapunov exponent $-\lambda$. The entry $b_n(x)(s_-(x) - s_{-n}(x)$, which represents the linearized dynamic along G_-, being multiplicative, we have along any μ-generic orbit : $-\lambda =$

$$\lim_{n\to\infty}\frac{1}{n}\log(b_n(x)(s_-(x) - s_{-n}(x)) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\log\left(b_1(f^k x)(s_-(f^k x) - s_{-1}(f^k x))\right)$$

In the same way, we have : $\lambda =$

$$\lim_{n\to\infty}\frac{1}{n}\log\left(b_n(x)(s_+(x)-s_{-n}(x))\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\log\left(b_1(f^kx)(s_+(f^kx)-s_{-1}(f^kx))\right)$$

By subtracting these two equalities and using Birkhoff's ergodic theorem, we obtain the equality of the theorem.

□

4. Invariant Lagrangian graphs : problems of C^1-regularity

In this section, we will explain the relation between the cones that are tangent to minimizing sets and the Green bundles. Then we will deduce some regularity results. In particular, we will improve Birkhoff's regularity result that asserts that any essential invariant curve of a twist map is a Lipschitz graph, and we will prove that C^0 integrability implies C^1 integrability on a dense G_δ-subset.

4.1. Two notions of C^1 regularity. We define what the "tangent vectors" to a set that is not necessarily a manifold are. This definition was given by G. Bouligand in the 30's in [12].

Definition. Let $K\subset\mathbb{T}^d\times\mathbb{R}^d$ be a subset of $\mathbb{T}^d\times\mathbb{R}^d$ and let $x\in K$ be a point of K. We say that $v\in T_x(\mathbb{T}^d\times\mathbb{R}^d)$ belongs to the contingent cone to K at x if there exist a sequence (x_n) of points of K converging to x and a sequence (t_n) of positive real numbers so that : $v=\lim\limits_{n\to\infty}t_n(x_n-x)$. The contingent cone to K at x is denoted by $\mathcal{C}_x(K)$.
We say that $v\in T_x(\mathbb{T}^d\times\mathbb{R}^d)$ belongs to the paratingent cone to K at x if two sequences (x_n) and (y_n) of points of K converging to x and a sequence (t_n) of positive real numbers exist so that : $v=\lim\limits_{n\to\infty}t_n(x_n-y_n)$. The paratingent cone to K at x is denoted by $\mathcal{P}_x(K)$.

As the sets in which we are interested are contained in some Lagrangian graphs, we give the following definitions of C^1 regularity :

Definition. We say that $K\subset\mathbb{T}^d\times\mathbb{R}^d$ is strongly C^1-regular at $x\in K$ if the paratingent cone $\mathcal{P}_x(K)$ is contained in a Lagrangian subspace.
We say that $K\subset\mathbb{T}^d\times\mathbb{R}^d$ is weakly C^1-regular at $x\in K$ if a Lagrangian subspace P of $T_x(\mathbb{T}^d\times\mathbb{R}^d)$ exists so that, for every sequence (x_n) of points of K converging to x, we have : $\limsup\limits_{n\to\infty}\mathcal{C}_{x_n}(K)\subset P$ where the lim sup is taken for the Hausdorff metric. The union of such limits will be called the generalized contingent cone and will be denoted by $\mathcal{C}_x^*(K)$.

Because $\mathcal{C}^*_x(K) \subset \mathcal{P}_x(K)$, strong C^1-regularity implies weak C^1-regularity. Moreover, if K is the graph of a Lipschitz map η above the zero section, then K is weakly C^1-regular if, and only if, K is strongly C^1-regular if, and only if, the map η is C^1.

4.2. Tangent cones and Green bundles. We recalled before that if $x \in \mathbb{T}^d \times \mathbb{R}^d$ is a point, there is a natural order between the Lagrangian subspaces of $T_x(\mathbb{T}^d \times \mathbb{R}^d)$ that are transverse to the vertical. But the tangent (contingent or paratingent) cone to a set is not necessarily contained in a linear Lagrangian subspace. We need, then, a new order to compare such a tangent cone to the Green bundles.

DEFINITION. If $A \subset T_x(\mathbb{T}^d \times \mathbb{R}^d)$ is a subset of $T_x(\mathbb{T}^d \times \mathbb{R}^d)$ an if P, P' are two Lagrangian subspaces of $T_x(\mathbb{T}^d \times \mathbb{R}^d)$ that are transverse to the vertical, we say that A is between P and P' and we write $P \leq A \leq P'$ if, for every $a \in A$, there exists a Lagrangian subspace P_A such that $P \leq P_A \leq P'$.

Let us notice that when A is a Lagrangian subspace, $P \leq A \leq P'$ is not ambiguous : it has the same meaning for the two orders. In the case of a twist map of the annulus, $P \leq A \leq P'$ just means that the slope of every element of A is between the slope of every element of P and the slope of every element of P'. Let us notice that : $P \leq A \leq P \Leftrightarrow A \subset P$. We prove this in [3] :

Theorem 8. *Let $f : \mathbb{A} \to \mathbb{A}$ be a positive twist map and K be an Aubry-Mather set. Then :*

$$\forall x \in K, G_-(x) \leq \mathcal{P}_x(K) \leq G_+(x).$$

Similarly, we have (a slightly different version of this is given in see [1]) :

Theorem 9. *Let $\mathcal{G}$ be the graph of a C^0 closed form η that is invariant by the Hamiltonian flow of the Tonelli Hamiltonian H. Then :*

$$\forall x \in \mathcal{G}, G_-(x) \leq \mathcal{P}_x(\mathcal{G}) \leq G_+(x).$$

Hence, for the C^0-Lagrangian graphs, the Green bundles give some bounds for the paratingent cones. We will see in section 7 that for the Aubry sets , we obtain a weaker result (which concerns only the generalized contingent cone); the reason is that in section 7, we use discontinuous Lagrangian graphs, called pseudo-graphs.

4.3. Regularity of invariant C^0-Lagrangian graphs. A classical result asserts that every C^0-Lagrangian invariant graph is locally minimizing. Then, we use two properties of the Green bundles that we found before : the dynamical criterion and the relation between the Green bundles and the paratingent cone to obtain some regularity results for the C^0-Lagrangian graphs (see [1], [4]). Let us mention that an invariant C^0-Lagrangian graph is always Lipschitz (see [19]), but it may happen that a Lipschitz graph is nowhere C^1.

At first, we obtain some results for small dimensions : in this case, the restricted linearized dynamic cannot tend to ∞ on a set with a non zero Lebesgue measure; hence, the two Green bundles are equal and the paratingent cone is a tangent subspace :

Theorem 10. *Let $f : \mathbb{A} \to \mathbb{A}$ be a twist map and let $\gamma : \mathbb{T} \to \mathbb{R}$ be a continuous map whose graph is invariant by f. Then there exists a dense G_δ subset U of $\mathbb{T}$ whose Lebesgue measure is 1 and so that every t of U is a point of differentiability of γ and a point of continuity of γ'. More precisely, the graph of γ is strongly C^1-regular at every point $(t, \gamma(t))$ with $t \in U$.*

This result improves G. Birkhoff's famous result asserting that such an invariant curve is always Lipschitz (see [11]) and proves that some Lipschitz graphs exist that are invariant by no twist map.
Some examples of twist maps exist that have such an invariant curve that is not C^1. But all the known examples have a rational rotation number. Hence, we ask :

Question 2. *Does an example of an invariant curve with an irrational rotation number that is not C^1 exist?*

Theorem 11. *Let $H : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a Tonelli Hamiltonian all of whose singularities are non-degenerate. Let $\mathcal{G}$ be a C^0- Lagrangian graph that is invariant by the Hamiltonian flow of H. If $\mathcal{G}$ is the graph of λ, then there exists a dense G_δ subset D of $\mathbb{T}^2$ with full Lebesgue measure so that λ is differentiable on D and its derivative is continuous at every point of D. More precisely, $\mathcal{G}$ is strongly C^1-regular at every $(q, \lambda(q))$ with $q \in D$.*

When we can specify the restricted dynamic in such a way that all the orbits of the restricted linearized dynamic are bounded, we have, too, $G_- = G_+$ and some regularity results :

Theorem 12. *Let $f : \mathbb{A} \to \mathbb{A}$ be a twist map and let $\gamma : \mathbb{T} \to \mathbb{R}$ be a continuous map whose graph is invariant by f. Let us assume that the restriction of f to the graph of γ is bi-Lipschitz conjugated to a rotation. Then γ is C^1 and the restriction of f to the graph of γ is C^1 conjugated to a rotation.*

Theorem 13. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and let $\mathcal{G}$ be a C^0-Lagrangian graph that is invariant by the Hamiltonian flow so that the time one flow restricted to $\mathcal{G}$ is bi-Lipschitz conjugated to a translation of $\mathbb{T}^d$. Then the graph $\mathcal{G}$ is C^1.*

Another interesting application of the Green bundles is a description of what happens in the C^0 completely integrable case :

Definition. Let $f : \mathbb{A} \to \mathbb{A}$ be a twist map. Then f is C^0-*integrable* if $\mathbb{A} = \bigcup_{\gamma \in \Gamma} G(\gamma)$ where :

1. Γ is a subset of $C^0(\mathbb{T}, \mathbb{R})$ and $G(\gamma)$ is the graph of γ;
2. $\forall \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2 \Rightarrow G(\gamma_1) \cap G(\gamma_2) = \emptyset$;
3. $\forall \gamma \in \Gamma, f(G(\gamma)) = G(\gamma)$.

REMARK. The general reference for this remark is [27].
A theorem of Birkhoff states that under the hypothesis of this definition, every $\gamma \in C^0(\mathbb{T}, \mathbb{R})$ whose graph is invariant by f is Lipschitz and that the set $\mathcal{L}(f)$ of those invariant graphs is closed for the C^0-topology.
If we fix a lift $\tilde{f}$ of f, we can associate with every $\gamma \in \mathcal{L}(f)$ its rotation number $\rho(\gamma)$. Then, if $\gamma_1, \gamma_2 \in \mathcal{L}(f)$, we have : $G(\gamma_1) \cap G(\gamma_2) \neq \emptyset \Rightarrow \rho(\gamma_1) = \rho(\gamma_2)$ and $G(\gamma_1) \cap G(\gamma_2) = \emptyset \Rightarrow \rho(\gamma_1) \neq \rho(\gamma_2)$. We deduce that $\mathcal{L}(f) = \Gamma$ and therefore Γ is closed for the C^0 topology.

Theorem 14. *Let $f : \mathbb{A} \to \mathbb{A}$ be a twist map that is C^0 integrable. Let Γ be the set of $\gamma \in C^0(\mathbb{T}, \mathbb{R})$ whose graph is invariant under f. Then a dense G_δ subset $\mathcal{G}$ of Γ endowed with the C^0-topology exists so that : every $\gamma \in \mathcal{G}$ is C^1. Moreover, in $\mathcal{G}$, the C^0-topology is equal to the C^1-topology.*

We say that a Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is C^0-integrable if there exists a partition $\mathcal{P}$ of $\mathbb{T}^d \times \mathbb{R}^d$ into C^0-Lagrangian graphs that are invariant by the flow and so that the map sending an element of $\mathcal{P}$ on its cohomology class sends $\mathcal{P}$ onto $H^1(M, \mathbb{R})$.

Theorem 15. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^0-integrable Tonelli Hamiltonian and let Λ_1 be the set of closed 1-forms whose graphs are elements of the partition. Then a dense G_δ subset of Λ_1 all of whose elements are C^1, exists.*

Proof. To prove the two last theorems, we begin by noticing that there is a dense set of completely periodic invariant Lagrangian graphs $\mathcal{G}$, i.e. so that there exists $T > 0$ satisfying : $\varphi_{T|\mathcal{G}} = Id_{\mathcal{G}}$. Using the dynamical criterion, on such a graph we have then $G_- = G_+$. Using the semicontinuity of G_- and G_+, we obtain $G_- = G_+$ on a dense G_δ-subset of invariant graphs, and then these invariant graphs are C^1. □

Question 3. *Are there examples of Tonelli Hamiltonians or twist maps that are C^0-integrable but not C^1-integrable?*

5. The link between the shape of the Aubry-Mather sets and their Lyapunov exponents

In this section, we give a complete characterization of the Aubry-Mather sets that are uniformly hyperbolic and of the minimizing measures with non-zero Lyapunov exponents. The following results are proved in [3].

Theorem 16. *Let f be a twist map and let M be an Aubry-Mather set of f with no isolated point. The two following assertions are equivalent :*
- *for all $x \in M$, M is not strongly C^1-regular at x;*
- *the set M is uniformly hyperbolic (for f).*

An amusing corollary is the following : if M is an Aubry-Mather set with no isolated point for two twist maps f_1 et f_2, then it is uniformly hyperbolic for f_1 if, and only if, it is hyperbolic for f_2.

Proof. For the direct sense, we know that the C^1-irregularity implies the transversality of the two Green bundles and then the uniform hyperbolicity. For the other sense, we prove that when M is uniformly hyperbolic, we have along M : $E^u \cup E^s \subset \mathcal{P}(M)$ and then the irregularity. □

Theorem 17. *Let f be a twist map and let μ be a minimizing measure whose support has no isolated point. The two following assertions are equivalent :*
- *for μ-almost every x, the support of μ, denoted by* $\text{supp}\mu$*, is C^1-regular at x;*
- *the Lyapunov exponents of μ (for f) are zero.*

Proof. The fact that irregularity implies non zero Lyapunov exponents is very similar to what was done in the proof of theorem 16. If now $\text{supp}\mu$ is C^1 regular μ-almost everywhere, the projected dynamic of $f_{|\text{supp}\mu}$ is C^1-conjugated to the initial dynamic $f_{|\text{supp}\mu}$ μ almost everywhere. We can extend this projected dynamic to a bi-Lipschitz homeomorphism h of $\mathbb{T}$. For such a bi-Lipschitz homeomorphism, we may define a kind of modified Lyapunov exponent and prove that it is zero everywhere by using a subtle improvement of Klingman's sub-multiplicative ergodic theorem due to A. Furman (see [24]). □

Hence, knowing the measure μ, we can say if the Lyapunov exponents are zero or not, but the knowledge of its support is a priori not sufficient to deduce if the Lyapunov exponents are zero or no. To be more precise, it is interesting to answer the following questions :

Question 4. *Do two twist maps f and g and two minimizing measures μ_f for f and μ_g for g exist, so that μ_f and μ_g have the same support but are not equivalent (i.e. not mutually absolutely continuous)?*

Another question concerns the existence of such non-uniformly hyperbolic measures :

Question 5. *Do there exist any minimizing measures with non zero Lyapunov exponents that are not uniformly hyperbolic?*

However, in extreme cases, we obtain a result concerning the link between the support and the Lyapunov exponents :

Corollary 18. *Let f be a twist map and let μ be a minimizing measure whose support has no isolated point. If the support is C^1-regular everywhere, then the Lyapunov exponents of μ are zero.*

It is not hard to see that an Aubry-Mather set is everywhere C^1-regular if, and only if, a C^1 map $\gamma : \mathbb{T} \to \mathbb{R}$ exists, whose graph contains M. In [27], M. Herman gives some examples of Aubry-Mather sets that are invariant by a twist map, contained in a C^1-graph but not contained in an *invariant* continuous curve.

Question 6. *Do any examples of minimizing measures with zero Lyapunov exponents that are not supported in a C^1 curve exist?*

It is possible that the numerical evidence contained in [6] and [7] gives such examples.

6. Weak KAM theory

In the case of a completely integrable Tonelli Hamiltonian, the manifold $\mathbb{T}^d \times \mathbb{R}^d$ is foliated by invariant Lagrangian tori that are graphs. When we perturb such a Hamiltonian, a lot of these tori persist, due to the strong KAM theorems.
The invariant "pseudographs" that we will study in this section are in a certain sense the ghosts of the invariant Lagrangian graphs. It may happen that a Tonelli Hamiltonian has no invariant Lagrangian graph, but it always has some negatively (resp. positively) invariant discontinuous Lagrangian graphs, called pseudographs, which contain true invariant subsets. This name of "pseudograph" is due to P. Bernard (see [8]) and the proof of the existence of negatively invariant pseudographs is what is called the weak KAM theorem and is due to A. Fathi ([18], [20]). We won't give any proof in this section, but all the results that we give here are proved in [18] or [8].

6.1. The Lax-Oleinik semigroup and its interpretation on pseudographs. Before explaining what a pseudograph is and which kind of transformation of these pseudographs to consider, let us define a semigroup on the set $C^0(\mathbb{T}^d, \mathbb{R})$ of continuous maps from $\mathbb{T}^d$ to $\mathbb{R}$. This semigroup is well-known in PDE.

The *negative semigroup* $(T_t^-)_{t>0}$ *of Lax-Oleinik* is defined on $C^0(\mathbb{T}^d, \mathbb{R})$ by :

$$T_t^- u(q_0) = \inf \left(u(q) + \int_0^t L(\gamma(s), \dot\gamma(s))ds \right);$$

where the infimum is taken on all the C^1 arcs $\gamma : [0,t] \to \mathbb{T}^d$ such that $\gamma(t) = q_0$.

In [18] (see also [8]), A. Fathi proves that for every $\varepsilon > 0$, there exists a constant $K > 0$ such that for every $t \geq \varepsilon$ and every $u \in C^0(\mathbb{T}^d, \mathbb{R})$, the function $T_t^- u$ is K-semi-concave where :

1. A function $v : V \to \mathbb{R}$ defined on a subset V of $\mathbb{R}^d$ is *K-semi-concave* if for every $x \in V$, there exists a linear form p_x defined on $\mathbb{R}^d$ so that :

   $$\forall y \in V, v(y) \leq v(x) + p_x(y-x) + K\|y-x\|^2.$$

   Then we say that p_x is a *K-super-differential* of v at x.

2. Let us fix a finite atlas $\mathcal{A}$ of the manifold $\mathbb{T}^d$; a function $u : \mathbb{T}^d \to \mathbb{R}$ is *K-semi-concave* if for every chart (U, ϕ) belonging to $\mathcal{A}$, $u \circ \phi^{-1}$ is K-semi-concave. Then a *K-super-differential* of u at q is $p_x \circ D\phi(q)$ where p_x is a K-super-differential of $u \circ \phi^{-1}$ at $x = \phi(q)$.

A semi-concave function is always Lipschitz and so differentiable almost everywhere and for such a function, we define its pseudograph :
a *pseudograph* is the graph $\mathcal{G}(du)$ of du, where $u : \mathbb{T}^d \to \mathbb{R}$ is a semi-concave function.

As the images of any continuous function by the Lax-Oleinik semigroup are semi-concave, we had better consider the action of the Lax-Oleinik group on the set $\mathcal{SC}(\mathbb{T}^d)$ of the semi-concave functions of $\mathbb{T}^d$.

A very nice interpretation of the action of the semigroup in terms of pseudographs is given in [8] :

Theorem 19. *(P. Bernard) Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian whose flow is denoted by (φ_t) and let $u : \mathbb{T}^d \to \mathbb{R}$ be a semi-concave function. Then :*

$$\forall t > 0, \overline{\mathcal{G}(dT_t^- u)} \subset \varphi_t(\mathcal{G}(du)).$$

Hence, if we are looking at the pseudographs, the action of T_t^- on $u \in \mathcal{SC}(\mathbb{T}^d)$ consists in cutting the image $\varphi_t(\mathcal{G}(du))$ of the pseudograph of du by the flow, removing some parts of this set to obtain a new pseudograph.

6.2. The weak KAM theorem and Mañé's critical value. The weak KAM theorem, due to A. Fathi, gives us some pseudographs that are invariant by this action :

Theorem 20 (Weak KAM theorem, A. Fathi). *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian. There exists a unique constant $c \in \mathbb{R}$ such that the modified semigroup $(\tilde{T}_t^-)$ defined by :*

$$\tilde{T}_t^- u = T_t^- u + ct$$

has at least one fixed point. Such a fixed point is called a weak KAM solution and c is called Mañé's critical value.

We denote by $\mathcal{S}^-(H)$ the set of weak KAM solutions for H. The weak KAM theorem proves the existence of some negatively invariant pseudographs such that : $\forall t > 0, \varphi_{-t}(\overline{\mathcal{G}(du)}) \subset \mathcal{G}(du)$. As said earlier, a compact invariant subset corresponds to such a pseudograph :

$$I(du) = \bigcap_{t>0} \varphi_{-t}(\mathcal{G}(du)) = \bigcap_{t>0} \varphi_{-t}(\overline{\mathcal{G}(du)}).$$

This set $I(du)$ is, in fact, a Lipschitz graph.
There is a relation between the Lax-Oleinik semigroup and the Hamilton-Jacobi equation :

Proposition 21. *(A. Fathi) Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and let $u : \mathbb{T}^d \to \mathbb{R}$ be a semi-concave function. Then u is a weak KAM solution associated with H if, and only if, it is a solution to the Hamilton-Jacobi equation $H = c$, i.e. if at every point of differentiability q of u, we have : $H(q, du(q)) = c$.*

Hence, the pseudographs of the weak KAM solutions are all contained in the critical level $\{H = c\}$.
Let us mention that the result given by A. Fathi is more general, because it deals with the viscosity solution of the Hamilton-Jacobi equation; we don't want to define this notion, but the reader can find some material in [15]. A good introduction to the PDE aspects of the weak KAM theory can be found in [17].

6.3. Mather, Aubry and Mañé sets. There is a third characterization of Mañé's critical value c :

$$-c = \inf_{\mu} \int_{\mathbb{T}^d\times\mathbb{R}^d} Ld\mu$$

where μ varies among the Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ that are invariant by the Euler-Lagrange flow of L . This lower bound is, in fact, achieved by a measure with compact support. A Borel probability measure μ with compact support in $\mathbb{T}^d \times \mathbb{R}^d$ is said to be minimizing if it is invariant by the Euler-Lagrange flow and satisfies $-c = \int_{\mathbb{T}^d\times\mathbb{R}^d} Ld\mu$. It can be proved that for ergodic measures, this definition is equivalent to the one that we gave before (via the Legendre map). If we denote by suppμ the support of the measure μ, the *Mather set* is defined by :

$$\mathcal{M}^*(H) = \bigcup_{\mu} \mathcal{L}^{-1}(\mathrm{supp}\mu).$$

J. Mather proved that $\mathcal{M}^*(H)$ is an invariant non-empty compact subset of $\mathbb{T}^d \times \mathbb{R}^d$ which is a Lipschitz graph above a compact part of the zero section. A. Fathi proved that the pseudograph of any weak KAM solution contains the Mather set. Moreover, for such a weak KAM solution u, any invariant Borel probability measure whose support is contained in $\mathcal{G}(du)$ is, in fact, the image via the Legendre map of a minimizing measure.

The Aubry set is defined by : $\mathcal{A}^*(H) = \bigcap_{u\in\mathcal{S}^-(H)} I(du)$ and the projected Aubry set is : $\mathcal{A}(H) = \pi(\mathcal{A}^*(H))$. The Aubry set is then an invariant compact Lipschitz graph above a part of the zero section.
The Mañé set is defined by :

$$\mathcal{N}^*(H) = \bigcup_{u\in\mathcal{S}^-(H)} I(du).$$

It is compact and invariant, but in general, it is not a graph.
We have : $\mathcal{M}^*(H) \subset \mathcal{A}^*(H) \subset \mathcal{N}^*(H) \subset \mathcal{E} = H^{-1}(c)$.

There are some other characterizations of the Aubry and Mañé sets (see [32] and [13]). Following Mañé, let us define the Mañé potential. For all $(q_1, q_2) \in M^2$ and all $t > 0$, we define : $a_t(q_1, q_2) = \inf \int_0^t L(\gamma(s), \dot\gamma(s))ds$ where the infimum is taken on all the C^1 curves $\gamma : [0, t] \to \mathbb{T}^d$ such that $\gamma(0) = q_1$ and $\gamma(t) = q_2$. The Mañé potential is defined by : $\Phi(q_1, q_2) = \inf_{t>0}(a_t(q_1, q_2) + ct)$. A curve $\gamma : I \to \mathbb{T}^d$ is said to be semi-static if for all $t_1 < t_2$ in I : $\int_{t_1}^{t_2}(L(\gamma(t),\dot\gamma(t))+c)dt = \Phi(\gamma(t_1),\gamma(t_2))$. Then a curve $\gamma : \mathbb{R} \to \mathbb{T}^d$ is semi-static if, and only if, $\mathcal{L}^{-1}(\gamma,\dot\gamma)$ is the orbit of a point of the Mañé set.

Following Mather (see [35]), we define the Peierls barrier : $h : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ by : $h(q_1,q_2) = \liminf_{t\to+\infty}(a_t(q_1,q_2)+ct)$. A. Fathi proved that, in fact, we have a true (and uniform) limit. Then a point q is in the projected Aubry set if, and only if, $h(q,q) = 0$. Moreover, if $\gamma_n : [0,t_n] \to \mathbb{T}^d$ is a sequence of arcs so that $\gamma_n(0) = \gamma_n(t_n) = q$, $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty}\int_0^{t_n}(L(\gamma_n(t),\dot\gamma_n(t))+c) = 0$, then $\lim_{n\to\infty}\mathcal{L}^{-1}(\gamma_n(t_n),\dot\gamma_n(t_n)) = \lim_{n\to\infty}\mathcal{L}^{-1}(\gamma_n(0),\dot\gamma_n(0)) = (q,p)$ where (q,p) is the point of the Aubry set so that $\pi(q,p) = q$.

6.4. The symmetrical Hamiltonian and the positive Lax-Oleinik semigroup. If $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Tonelli Hamiltonian, its symmetrical Hamiltonian is defined by : $\check H(q,p) = H(q,-p)$. Then its associated Lagrangian is defined by : $\check L(q,v) = L(q,-v)$. If (φ_t) (resp. $(\check\varphi_t)$) is the Hamiltonian flow associated with H (resp. $\check H$), if we denote by $i : \mathbb{T}^d\times\mathbb{R}^d \to \mathbb{T}^d\times\mathbb{R}^d$ the involution : $i(q,p) = (q,-p)$, then we have : $\check\varphi_t(q,p) = i\circ\varphi_{-t}\circ i(q,p)$. Moreover, $\gamma : \mathbb{R}\to\mathbb{T}^d$ is a solution to the Euler-Lagrange equations for L if, and only if, $\check\gamma : \mathbb{R}\to\mathbb{T}^d$ defined by $\check\gamma(t) = \gamma(-t)$ is a solution to the Euler-Lagrange equations for $\check L$ and $\gamma : [a,b]\to\mathbb{T}^d$ minimizes the Lagrangian action of L between $\gamma(a)$ and $\gamma(b)$ if, and only if, $\check\gamma : t\in[-b,-a]\to\gamma(-t)\in\mathbb{T}^d$ minimizes the action of $\check L$ between $\gamma(b)$ and $\gamma(a)$. From this remark, we deduce the following expression of the negative Lax-Oleinik semigroup of $\check L$:

$$\check T_t^- u(q) = \inf_\gamma\left(u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))ds\right)$$

where the infimum is taken on all the C^1 curves $\gamma : [0,t]\to\mathbb{T}^d$ such that $\gamma(0) = q$. We then define the positive Lax-Oleinik semigroup for H by : $T_t^+u(q) = -\check T_t^-(-u)$. Hence :

$$T_t^+u(q) = \sup_\gamma\left(u(\gamma(t)) - \int_0^t L(\gamma(s),\dot\gamma(s))ds\right).$$

Instead of restricting ourselves to the set of semi-concave functions, we now use the set of semi-convex functions (a function $u : \mathbb{T}^d\to\mathbb{R}$ is K-semi-convex if $-u$ is K-semi-concave). The graph of du where u is semi-convex is called an anti-pseudograph, and the anti-pseudograph $\mathcal{G}(du)$ of any fixed point u of the positive Lax-Oleinik semigroup of H satisfies : $\forall t>0, \overline{\mathcal{G}(du)}\subset\varphi_{-t}(\mathcal{G}(du))$.

Let us notice that H and $\check H$ have the same critical value (use the characterization by the minimizing measures) and that $\mathcal{M}^*(H) = i\left(\mathcal{M}^*(\check H)\right)$ is contained in the pseudograph of any positive weak KAM solution (for H). Moreover, A. Fathi proved that for any negative weak KAM solution u_-, there exists a unique positive weak KAM solution u_+ such that $u_{-|\mathcal{M}(H)} = u_{+|\mathcal{M}(H)}$. Such a pair (u_-,u_+) of weak KAM solutions with $u_{-|\mathcal{M}(H)} = u_{+|\mathcal{M}(H)}$ is called a pair of conjugate weak KAM solutions. We always have :

$u_+ \leq u_-$; $\pi(I(du_-)) = \{q \in M; u_-(q) = u_+(q)\}$; $du_{-|\pi(I(du_-))(H)} = du_{+|\pi(I(du_-))}$. A consequence is that, for every pair (u_-, u_+) of conjugate weak KAM solutions, the Aubry set (and then the Mather set) of H is in $\mathcal{G}(du_-) \cap \mathcal{G}(du_+)$.

7. Weak KAM solutions and Green bundles

We proved in [2] that the Green bundles give some bounds for the "second derivative" of the weak KAM solutions along the Aubry set; what we denote by $\tilde{G}_\pm$ is a modified Green bundle that is very close to the original Green bundle :

Theorem 22. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and let (u_-, u_+) be a pair of conjugate weak KAM solutions. Then :*

$$\forall x \in I(du_-), \tilde{G}_-(x) \leq \mathcal{C}_x(\mathcal{G}(du_-)) \leq G_+(x) \quad \text{and} \quad G_-(x) \leq \mathcal{C}_x(\mathcal{G}(du_+)) \leq \tilde{G}_+(x).$$

Proof. The proof is rather technical. We begin by selecting a pseudograph in the images of the physical verticals, where the physical vertical at $x \in \mathbb{T}^d \times \mathbb{R}^d$ is : $\mathcal{V}(x) = \pi^{-1}(\pi(x)) \subset \mathbb{T}^d \times \mathbb{R}^d$. Let $x_0 = (q_0, p_0) \in I(du_-)$. Then for every $t > 0$ we prove that there there exist two C^2- functions $g_t^+, g_t^- : \mathbb{T}^d \to \mathbb{R}$, the first one semi-concave and the second one semi-convex, so that $g_t^-(q_0) = u_-(q_0) = u_+(q_0) = g_t^+(q_0)$, $g_t^- \leq u_+ \leq u_- \leq g_t^+$ and : $\mathcal{G}(dg_t^+) \subset \varphi_t(\mathcal{V}(\varphi_{-t}x))$, $\mathcal{G}(dg_t^-) \subset \varphi_{-t}(\mathcal{V}(\varphi_t x))$. Then we manage to deduce that $\mathcal{C}_x(\mathcal{G}(du_-)) \leq G_t(x)$ and $G_{-t}(x) \leq \mathcal{C}_x(\mathcal{G}(du_+))$ where $G_t(x)$ (resp. $G_{-t}(x)$) is the tangent subspace at x to $\mathcal{G}(dg_t^+)$ (resp. $\mathcal{G}(dg_t^-)$). When t tends to $+\infty$, we find the results of the theorem. □

Corollary 23. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and let (u_-, u_+) be a pair of conjugate weak KAM solutions. Then :*

$$\forall x \in I(du_-), \tilde{G}_-(x) \leq \mathcal{C}_x(I(du_-)) \leq G_+(x)$$

Using the results of section 3 and the fact that the support of every minimizing measure is contained in a subset $I(du_-)$, we deduce :

Corollary 24. *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and let μ be a minimizing measure all of whose Lyapunov exponents are zero. Then, the support of μ is weakly C^1-regular at μ-almost every point.*

This last result is less complete than the one contained in theorem 17, because we obtain only one implication. In fact, the other implication is not correct in this case : it may happen that a hyperbolic set is very smooth, and then that a hyperbolic measure has a regular support.

References

[1] M.-C. Arnaud, *Fibrés de Green et régularité des graphes C^0-Lagrangiens invariants par un flot de Tonelli*, Ann. Henri Poincaré **9** (2008), no. 5, 881–926.

[2] M.-C. Arnaud, *Green bundles, Lyapunov exponents and regularity along the supports of the minimizing measures* , preprint arXiv:1003.2139

[3] M.-C. Arnaud, *The link between the shape of the Aubry-Mather sets and their Lyapunov exponents*, preprint arXiv:0902.3266

[4] M.-C. Arnaud, *Three results on the regularity of the curves that are invariant by an exact symplectic twist map*, Publ. Math. Inst. Hautes Etudes Sci. No. **109** , 1–17 (2009).

[5] S. Aubry & P. Y. Le Daeron. *The discrete Frenkel- Kontorova model and its extensions.* I. Exact results for the ground-states. Phys. D 8 (1983), no. 3, 381–422.

[6] C. Baesens & R. S. MacKay, *Cantori for multi-harmonic maps*, Physica D **69** (1993) 59–76

[7] C. Baesens & R. S. MacKay, *The one-to-two hole transition for cantori*, Physica D **71** (1994) 372–89

[8] P. Bernard. *The dynamics of pseudographs in convex Hamiltonian systems.* J. Amer. Math. Soc. 21 (2008), no. 3, 615–669.

[9] M. L Bialy & R. S. MacKay, *Symplectic twist maps without conjugate points.* Israel J. Math. **141**, 235–247 (2004).

[10] J. Bochi & M. Viana, *Lyapunov exponents: how frequently are dynamical systems hyperbolic?* Modern dynamical systems and applications, 271–297, Cambridge Univ. Press, Cambridge, 2004.

[11] G. D. Birkhoff, *Sur quelques courbes fermées remarquables*, Bull. Soc. Math. France. 80 (1932), 1-26

[12] G. Bouligand. *Introduction à la géométrie infinitésimale directe* (1932) Librairie Vuibert, Paris.

[13] G. Contreras & R. Iturriaga, *Global minimizers of autonomous Lagrangians*, preprint (2000), 208 pp (enlarged version of 22nd Brazilian Mathematics Colloquium (IMPA), Rio de Janeiro, 1999. 148 pp) : www.cimat.mx/ gonzalo/

[14] G. Contreras & R. Iturriaga, *Convex Hamiltonians without conjugate points*, Ergodic Theory Dynam. Systems **19** (1999), no. 4, 901–952.

[15] M. G. Crandall & P. L. Lions, *Viscosity solutions of Hamilton Jacobi equations.* Trans. Amer. Math. Soc. 277 (1983), no. 1, 1–42.

[16] P. Eberlein, *When is a geodesic flow of Anosov type?* I,II. J. Differential Geometry 8 (1973), 437–463; ibid. 8 (1973), 565–577

[17] L. C. Evans, *A survey of partial differential equations methods in weak KAM theory.* Comm. Pure Appl. Math. 57 (2004), no. 4, 445–480.

[18] A. Fathi, *Weak KAM theorems in Lagrangian dynamics*, book in preparation.

[19] A. Fathi, *Regularity of C^1 solutions of the Hamilton-Jacobi equation.* Ann. Fac. Sci. Toulouse Math. (6) **12** no. 4, 479–516 (2003).

[20] A. Fathi, *Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens.* C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 9, 1043–1046.

[21] P. Foulon, *Estimation de l'entropie des systèmes lagrangiens sans points conjugués* Ann. Inst. H. Poincaré Phys. Théor. **57** , no. 2, 117–146 (1992).

[22] J. Franks & C. Robinson, *A quasi-Anosov diffeomorphism that is not Anosov.* Trans. Amer. Math. Soc. 223 (1976), 267–278.

[23] A. Freire & R. Mañé. *On the entropy of the geodesic flow in manifolds without conjugate points.* Invent. Math. 69 (1982), no. 3, 375–392.

[24] A. Furman *On the multiplicative ergodic theorem for uniquely ergodic systems.* Ann. Inst. H. Poincaré Probab. Statist. 33 (1997), no. 6, 797–815.

[25] C. Golé, *Symplectic twist maps. Global variational techniques.* Advanced Series in Nonlinear Dynamics, 18. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xviii+305

[26] L. W. Green, *A theorem of E. Hopf* Michigan Math. J. **5** 31–34 (1958).

[27] M. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, Vol. 1, Asterisque **103-104** (1983).

[28] M. Herman, *Inégalités "a priori" pour des tores lagrangiens invariants par des difféomorphismes symplectiques.* Inst. Hautes Etudes Sci. Publ. Math. No. 70 (1989), 47–101 (1990).

[29] W. Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type.* Ann. of Math. (2) 99 (1974), 1–13.

[30] P. Le Calvez, *Les ensembles d'Aubry-Mather d'un difféomorphisme conservatif de l'anneau déviant la verticale sont en général hyperboliques.* C. R. Acad. Sci. Paris Sr. I Math. 306 (1988), no. 1, 51–54.

[31] R. Mañé. *Quasi-Anosov diffeomorphisms and hyperbolic manifolds.* Trans. Amer. Math. Soc. 229 (1977), 351–370.

[32] R. Mañé *Lagrangian flows: the dynamics of globally minimizing orbits.* International Conference on Dynamical Systems (Montevideo, 1995), 120–131, Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996.

[33] R. Mañé, *Generic properties and problems of minimizing measures of Lagrangian systems*, Nonlinearity **9** (1996), no. 2, 273–310.

[34] J. N. Mather. *Existence of quasiperiodic orbits for twist homeomorphisms of the annulus.*Topology 21 (1982), no. 4, 457–467.

[35] J. N. Mather. *Action minimizing invariant measures for positive definite Lagrangian systems* Math. Z. 207 (1991), no. 2, 169–207.

[36] J. N. Mather. *Variational construction of connecting orbits.* Ann. Inst. Fourier (Grenoble) 43 (1993), no. 5, 1349–1386.

[37] V. I. Oseledec *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems.* (Russian) Trudy Moskov. Mat. Obsc. 19 1968 179–210.

[38] C. Robinson, *A quasi-Anosov flow that is not Anosov.* Indiana Univ. Math. J. 25 (1976), no. 8, 763–767.

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