Hyperbolic periodic orbits and Mather sets in certain symmetric cases

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Abstract

We consider a C^∞ Lagrangian function $L : T^*M \to \mathbb{R}$ which is superlinear and convex in the fibers and has one antisymplectic symmetry. We prove that:

- in every energy level strictly above the critical one, there exists a Mather set which is the union of some periodic orbits;
- for L generic, these orbits are hyperbolic;
- on the torus, these orbits have one homoclinic orbit.

1 Introduction

Let M be a compact and connected manifold endowed with a Riemannian metric. We will denote by (x, v) a point of the tangent bundle TM with $x \in M$ and v a vector tangent at x. The projection $\pi : TM \to M$ is then $(x, v) \to x$. The notation (x, p) will designate a point of the cotangent bundle T^*M with $p \in T^*_xM$. and $\pi^* : T^*M \to M$ will be the canonical projection $(x, p) \to x$.

We consider a Lagrangian function $L : TM \to \mathbb{R}$ which is C^∞ and:

- uniformly superlinear: uniformly on $x \in M$, we have: $\lim\limits_{\|v\|\to+\infty} \dfrac{L(x, v)}{\|v\|} = +\infty$;

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- strictly convex: for all $(x,v)\in TM$, $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite.

We can associate to such a Lagrangian function the Legendre map $\mathcal{L}:TM\to T^*M$ defined by: $\mathcal{L}(x,v)=\frac{\partial L}{\partial v}(x,v)$ which is a fibered C^∞ diffeomorphism and the C^∞ Hamiltonian function $H:T^*M\to\mathbb{R}$ defined by: $H(x,p)=p\,(\mathcal{L}^{-1}(x,p))-L(\mathcal{L}^{-1}(x,p))$. The Hamiltonian function H is then superlinear, strictly convex in the fiber and C^∞. We denote by (f_t^L) the Euler-Lagrange flow associated to L and (ϕ_t^H) the Hamiltonian flow associated to H; then: $\phi_t^H=\mathcal{L}\circ f_t^L\circ\mathcal{L}^{-1}$.

In 1996, R. Mañé introduced the notion of "generic Lagrangian function " in [8]: *"a certain property holds for a generic Lagrangian L if, given a strictly convex and superlinear Lagrangian L_0, there exists a residual subset $\mathcal{O}\subset C^\infty(M)$ such that the given property holds for every Lagrangian L of the form $L=L_0+\psi$, $\psi\in\mathcal{O}$";*
and asked the following question (we will explain the notion of minimizing measure later): *"is it true that for a generic Lagrangian L, there exists a unique minimizing measure and this measure is supported by a periodic orbit?"*

He gave in [9] a partial answer to this question: *"A generic Lagrangian has a unique minimizing measure and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic".*

The first part of this result is proved in [8] and the last part in [2].

Finding hyperbolic periodic orbits for a Lagrangian function is an old and interesting problem, which is not necessarily related to the Mather set of a Lagrangian function (the Mather set is the image under the Legendre map of the closure of the union of the supports of the minimizing measures)[1]. In general, such periodic orbits are found by a minimizing method, but minimizing periodic orbits are not always hyperbolic; for example:

1. in [12], H. Poincaré proves that on every surface, every minimizing periodic orbit with one Floquet multiplier different from 1 is hyperbolic;

2. in [11], an example of a minimizing elliptic periodic orbit due to V. Bangert is given for a certain geodesic flow;

3. in [11], D. Offin proves that in a certain symmetric case for geodesic flows, some minimizing periodic orbits are hyperbolic or degenerate (with 4 Floquet multipliers equal to 1).

[1] In fact, what we called in this paper the "Mather set" and the "Aubry set" are usually called the "dual Mather set" and the "dual Aubry set", i.e. are the image under the Legendre map of the usual Mather set and Aubry set.

The first goal of this paper is to exhibit in a symmetric case (the symmetry is similar to the one used by Offin in [11] but the Lagrangian function is not necessarily geodesic) a relation between some minimizing periodic orbits and some Mather set. There are two reasons to do that:

1. this will allow us to prove in a "symmetric generic case" the existence of hyperbolic orbits by using a minimum of technical computations;

2. for a "generic symmetric" Lagrangian function defined on the torus $\mathbb{T}^n$, this implies the existence of homoclinic points for this periodic orbit.

Let us now explain which hypothesis we use:

(a) M is a closed connected manifold and $s : M \to M$ is a C^∞- diffeomorphism which is an involution: $s^2 = Id_M$ such that the set $\text{Fix}(s)$ of fixed points of s is a (non-empty) hypersurface $\mathcal{F}$ of M; moreover, we impose an extra topological assumption: one of the connected components C of $M\backslash\mathcal{F}$ can be completed in a compact manifold $\hat{C}$ with boundary such that the closure of C in $\hat{C}$ is $\hat{C}$ and the boundary of $\hat{C}$ has at least two connected components[2];

(b) $L : TM \to \mathbb{R}$ is a C^∞ Lagrangian function which is superlinear, strictly convex and has the following symmetry: $\forall (x, v) \in TM, L(s(x), -Ds(x)v) = L(x, v)$. We will say that this Lagrangian function is "s-symmetric."

For Lagrangian functions satisfying (a) and (b), a property will be generic (we will say "s-generic") if for any s-symmetric Lagrangian function L_0 satisfying (a) and (b), there exists a dense G_δ subset $\mathcal{G}$ of the set $C_s^\infty(M, \mathbb{R}) = \{V \in C^\infty(M, \mathbb{R}); V \circ s = V\}$ of s-symmetric functions such that for every $\psi \in \mathcal{G}$, the property holds for $L_0 + \psi$.

When $L : TM \to \mathbb{R}$ is a C^∞ Lagrangian function which is superlinear, strictly convex, we can associate to L the so-called "critical value" $c(L)$ of L which is the minimum of the $c \in \mathbb{R}$ such that $\{H \leq c\}$ contains an exact Lagrangian C^0-graph, i.e. the graph of du where $u : M \to \mathbb{R}$ is C^1: there exists many characterizations of this critical value and the one we use is studied in [7].

The results which we obtain are then ($(x, p) : \mathbb{R} \to T^*M$ is s-symmetric if $\forall t \in \mathbb{R}, (s \circ x(t), -p(t) \circ Ds(x)^{-1}) = (x(-t), p(-t))$):

[2]in fact we will see that $M\backslash\mathcal{F}$ has one or two connected components and that when there are two components, they are diffeomorphic.

Theorem 1 *If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ satisfy the hypothesis (a), (b) and if $c > c(L)$, there exists a s-symmetric closed 1-form λ of M such that the Mather set of the Lagrangian function $L+\lambda$ is contained in $\{(x,p) \in T^*M; H(x, p-\lambda(x)) = c\}$ and is a union of non critical and s-symmetric periodic orbits for the Hamiltonian function associated to the Lagrangian function $L + \lambda$.*

Theorem 2 *If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ s-generic satisfy the hypothesis (a), (b), there exists a dense set $D(L)$ of $]c(L), +\infty[$ such that for every $c \in D(L)$, there exists a s-symmetric closed 1-form λ of M such that the Mather set of the Lagrangian function $L+\lambda$ is a non critical s-symmetric periodic hyperbolic orbit contained in $\{(x,p) \in T^*M; H(x, p - \lambda(x)) = c\}$.*

Corollary 3 *If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ s-generic satisfy the hypothesis (a), (b), there exists a dense open set $U(L)$ of $]c(L), +\infty[$ such that for every $c \in U(L)$, $\{H = c\}$ contains one hyperbolic non critical periodic orbit.*

When the manifold is the torus, we obtain:

Corollary 4 *Let us assume that $n \geq 2$. If $s : \mathbb{T}^n \to \mathbb{T}^n$ and $L : TM \to \mathbb{R}$ s-generic satisfy the hypothesis (a), (b) and if $c(L)$ is the critical value of L, there exists a dense subset $D(L)$ of $]c(L), +\infty[$ such that for every $c \in D(L)$, $\{H = c\}$ contains one hyperbolic non critical periodic orbit which has one homoclinic orbit.*

Example : let $g : T\mathbb{T}^n \to T\mathbb{T}^n$ be a Riemannian metric such that: $\forall(\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{T}^n, \forall(v_1, \dots, v_n) \in \mathbb{R}^n, g(\theta_1, \theta_2, \dots, \theta_n)(v_1, \dots, v_n) = g(-\theta_1, \theta_2, \dots, \theta_n)(v_1, \dots, v_n)$ (for example the flat metric). Then there exists a dense G_δ subset $\mathcal{G}$ of $C^\infty_s(\mathbb{T}^n) = \{V : \mathbb{T}^n \to \mathbb{R}; \forall(\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{T}^n, V(\theta_1, \theta_2, \dots, \theta_n) = V(-\theta_1, \theta_2, \dots, \theta_n)\}$ such that, if $V \in \mathcal{G}$, and $m(V) = \max V$, there exists a dense subset, $D(V)$ of $]m(V), +\infty[$ such that for every $c \in D(V)$, there exists one hyperbolic orbit for the Lagrangian $g - V$ in $\{g + V = c\}$ having a homoclinic point. We have an analogous result for the symmetry: $s(\theta_1, \theta_2, \theta_3, \dots, \theta_n) = s(\theta_2, \theta_1, \theta_3, \dots, \theta_n)$.

Before proving these results, let us explain why, in the geodesic case, this system is a "billiard-like system". We remark (this is proved in Appendix A):

Proposition 5 *Let $s : M \to M$ be a C^∞- diffeomorphism which is an involution: $s^2 = Id_M$ such that the set $\mathrm{Fix}(s)$ of fixed points of s is a (non-empty) hypersurface $\mathcal{F}$ of M; then for every $x \in \mathcal{F}$, $Ds(x)$ is a reflexion and $M \backslash \mathcal{F}$ has at most two connected components.*

Let us notice that it may happen that $M\backslash\mathcal{F}$ has exactly one connected component: see the appendix A for an example.

Let $L = g$ be a s-symmetric Riemannian metric with geodesic flow (f_t) and let $\mathcal{F}$ be the hypersurface of fixed points of s. Let B be one of the connected components of $M\backslash\mathcal{F}$ and let $\bar{B}$ be the corresponding compact manifold with boundary (if $M\backslash\mathcal{F}$ has exactly one connected component, this manifold is not the closure of B in M(in this case, the closure of B in M is M)) such that the interior of $\bar{B}$ is B. We will denote by ∂B the boundary of $\bar{B}$. For every $x \in \partial B$, u_x is the eigenvector of $Ds(x)$ for the eigenvalue -1 such that $g(u_x) = 1$ and the geodesic half-line corresponding to u_x begins in B (i.e. u_x is oriented towards B). Let us notice that $g \circ s = g$ and thus $Ds(x)$ is an orthogonal reflexion for g and u_x is in the orthogonal line to $T_x\mathcal{F}$ for g.

We define a bundle $V(\partial B)$ over ∂B: if $x \in \partial B$, if $v \in T_xM$ such that $g(v) = 1$ is: $v = \lambda_v.u_x + w$ with $w \in T_x\mathcal{F}$, then: $(x \in V(\partial B) \iff \lambda_v > 0)$. Alternatively:

$$\forall x \in \partial B, (v \in V(\partial B)_x) \iff (g(v) = 1 \quad \text{and} \quad g(v, u_x) > 0).$$

We define $V'(\partial B)$ as being the set of $v \in V(\partial B)$ such that the geodesic half-line corresponding to v is transverse to $\mathcal{F}$ (for every time) and meets $\mathcal{F}$ infinitely many times.
If $v \in V'(\partial B)$, for small positive times, the geodesic half-line corresponding to v is in B; we follow this geodesic in B until it meets ∂B for the first time at $v_1 = b_1(v) \in T_{x_1}(\partial B)$. If we define: $b(v) = Ds(x_1)v_1 = Ds \circ b_1(v)$, then b is a billiard map.

Let us now compare the billiard map b with the map P which is the first return Poincaré map on $TM_{|\partial B}$. By definition of b:

1. if $v \in V'(\partial B)$, $P(v) = Ds \circ b(v)$ and $-P(v) \in V'(\partial B)$;

2. as g is s-symmetric: if $-v \in V'(\partial B)$, $P(v) = b \circ Ds(v)$ and $P(v) \in V'(\partial B)$.

Thus for every $v \in V'(\partial B)$ and $n \in \mathbb{N}$: $P^{2n}(v) = b^{2n}(v)$. In this sense the Poincaré map P represents a "billiard-like" map.

2 Proof of theorem 1

2.1 Aubry set and critical value

When $L : TM \to \mathbb{R}$ is a C^∞ Lagrangian function which is superlinear and strictly convex, we associate to L its critical value $c(L)$ which is the minimum of the $c \in \mathbb{R}$ such that $\{H \le c\}$ contains an exact Lagrangian C^0-graph, i.e. the graph of du where $u : M \to \mathbb{R}$

is C^1. We call the corresponding energy level $\{H = c(L)\}$ the critical level (there exists many characterizations of the critical level, we use the one of [7]). If $c \geq c(L)$ and if the graph $G(du)$ of the derivative du of a C^1-function $u : M \to \mathbb{R}$ is in $\{H \leq c\}$, we say that u is a c-subsolution for L and if $c = c(L)$, we say that u is a critical subsolution. Therefore a c-subsolution is a C^1 function $u : M \to \mathbb{R}$ such that the graph of du is contained in the complement of the unbounded connected component of $M \backslash \{H = c\}$. We say that the subsolution is strict if its graph is contained in $\{H < c\}$. Then:

Proposition 6 *Let L_1 and L_2 be two C^∞ Lagrangian functions which are superlinear and strictly convex. Let H_1 and H_2 be the associated Hamiltonian functions. Then, if the critical level $\{H_1 = c(L_1)\}$ of H_1 is one energy level of H_2, it is the critical level of H_2.*

In other words, given a level set of a Hamiltonian function associated to a C^∞ Lagrangian function which is superlinear and strictly convex, we don't need to know the considered Lagrangian (or Hamiltonian) function in order to determinate if this level is critical or not. The reason (and the proof) of this result is only the geometric characterization of the critical level with the subsolutions.

In the critical level $\mathcal{H}$ of such a Lagrangian function L, there exists a particular compact subset $\mathcal{A}(L)$, named the "Aubry set" of L. This set is a non empty, compact and Lipschitz graph above a part of the zero-section (see [6] or [9]), and one of its characterizations is (see [7]):

1. for every C^1 critical subsolution $u : M \to \mathbb{R}$, one has : $\mathcal{A}(L) \subset G(du)$;
2. there exists a C^1 critical subsolution $u : M \to \mathbb{R}$ is such that $\mathcal{A}(L) = G(du) \cap \mathcal{H}$.

Proposition 7 *Let L_1 and L_2 be two C^∞ Lagrangian functions which are superlinear and strictly convex. Let H_1 and H_2 be the associated Hamiltonian functions. We assume that H_1 and H_2 have the same critical level set, denoted $\mathcal{H}$. Then they have the same Aubry set: $\mathcal{A}(L_1) = \mathcal{A}(L_2)$.*

In other words, given the critical level set $\mathcal{H}$ of a Hamiltonian function associated to a C^∞ Lagrangian function which is superlinear and strictly convex, we don't need to know the considered Lagrangian (or Hamiltonian) functions in order to determinate its Aubry set. The reason (and the proof) of this result is only the geometric characterization of the Aubry set with the critical subsolutions.

Let us assume now that L satisfies the hypothesis (a) and (b): then L is s-symmetric and we have[3]: $\forall (x,p) \in T^*M, H(s(x), -p \circ (Ds(x))^{-1}) = H(x,p)$. Therefore (see appendix B), if $u : M \to \mathbb{R}$ is a C^1 c-subsolution, $-u \circ s$ is a c-subsolution too. The Hamiltonian function H being convex in the fiber, the averaged function $U = \frac{u - u\circ s}{2}$ is a c-subsolution which satisfies: $U \circ s = -U$. We have then:

Proposition 8 *If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ satisfy the hypothesis (a), (b), there exists a C^1 critical subsolution u such that $u \circ s = -u$ and $\mathcal{A}(L) = G(du) \cap \mathcal{H}$. Moreover, if $c > c(L)$, there exists a C^∞ strict c-subsolution u such that $u \circ s = -u$.*

The first part of this proposition is a consequence of the existence of a C^1 critical subsolution characterizing the Aubry set and of the convexity of H, and the last part is a consequence of the existence, for every $c > c(L)$, of a C^∞ strict c-subsolution (see [3]) and of the convexity of H.

Corollary 9 *If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ satisfy the hypothesis (a), (b), then:*

$$\{(s(x), -p \circ (Ds(x))^{-1}); (x,p) \in \mathcal{A}(L)\} = \mathcal{A}(L).$$

2.2 Construction of an auxiliary homogeneous Hamiltonian function

In this section, we consider a C^∞ Lagrangian function L which is superlinear, strictly convex and s-symmetric and $c > c(L)$. By proposition 8, there exists a C^∞ strict c-subsolution $u : M \to M$ such that: $u \circ s = -u$. Then: when L is a C^∞ Lagrangian function which is superlinear, strictly convex, s-symmetric and has a s-symmetric c-subsolution $u : M \to \mathbb{R}$, if H is the associated Hamiltonian function, we can construct a "simpler" Hamiltonian function $H_0 = H_0(u,c) : T^*M \to \mathbb{R}$ such that $\{H_0 = 1\} = \{H = c\}$:

Proposition 10 *Let L be a C^∞ Lagrangian function which is superlinear, strictly convex, s-symmetric and let $u : M \to \mathbb{R}$ be a C^∞ strict c-subsolution such that: $u \circ s = -u$. We denote by H the Hamiltonian function associated to L . Then there exists a unique Hamiltonian function $H_0 : T^*M \to \mathbb{R}$ such that:*

1. *$\{H_0 = 1\} = \{H = c\}$;*
2. *for every $x \in M$, the function $p \in T_x^*M \to H_0(p + du(x))$ is positively homogeneous with degree 2: $\forall p \in T_x^*M, \forall \lambda \geq 0, H_0(\lambda p + du(x)) = \lambda^2 H_0(p + du(x))$.*

[3]This will be proved in appendix B.

Then H_0 is superlinear, strictly convex and C^∞ on the complement of the graph of du and C^1 on T^*M. Moreover: $\forall (x,p) \in T^*M, H_0(s(x), -p \circ Ds(x)^{-1}) = H_0(x,p)$ and there exists a Riemannian metric g on M such that: $\forall (x,p) \in T^*M, g(p) \leq H_0(du(x)+p)$.

The interest of this new Hamiltonian function is:

1. we have $\{H_0 = 1\} = \{H = c\} = \mathcal{H}$, then the Hamiltonian flow $(\phi_t^{H_0})$ restricted to $\mathcal{H}$ is a reparametrization of the Hamiltonian flow (ϕ_t^H) restricted to $\mathcal{H}$;

2. $p \to H_0(p + du(x))$ is homogeneous in the fiber; therefore for every $h > 0$ the Hamiltonian flow $(\phi_t^{H_0})$ restricted to $\mathcal{H}$ is C^∞- conjugate to a reparametrization of the Hamiltonian flow $(\phi_t^{H_0})$ restricted to $\{H_0 = h\}$.

More precisely, for such a Hamiltonian function , we have:

Proposition 11 *Let $H_0 : T^*M \to \mathbb{R}$ be a Hamiltonian function satisfying the hypothesis (a), (b) such that there exists a C^∞ function $u : M \to \mathbb{R}$ such that for every $x \in M$, the function $p \in T_x^*M \to H_0(p + du(x))$ is positively homogeneous with degree 2.*

Let λ be a closed 1-form such that $\mathcal{A}(L_0 + \lambda) \subset \{(x,p) \in T^*M; H_0(x, p - \lambda(x)) = C\}$ with $C > 0$. Then $\mathcal{A}(L_0 + \frac{1}{\sqrt{C}}\lambda) \subset \{(x,p) \in T^*M; H_0(x, p - \frac{1}{\sqrt{C}}\lambda(x)) = 1\}$ and:

- *$\forall (x,p) \in T^*M, H_0(x, du(x) + p) = C \Leftrightarrow H_0(x, du(x) + \frac{1}{\sqrt{C}}p) = 1$;*
- *$\forall (x,p) \in T^*M, (x, du(x) + p + \lambda(x)) \in \mathcal{A}(L_0 + \lambda) \subset \{(x,p) \in T^*M; H_0(x, p - \lambda(x)) = C\} \Leftrightarrow (x, du(x) + \frac{1}{\sqrt{C}}p + \frac{1}{\sqrt{C}}\lambda(x)) \in \mathcal{A}(L_0 + \frac{1}{\sqrt{C}}\lambda) \subset \{(x,p) \in T^*M; H_0(x, p - \frac{1}{\sqrt{C}}\lambda(x)) = 1\}$;*
- *If $H_0(x, du(x) + p) = C$ with $C > 0$ and if we write: $\phi_t^{H_0}(x, du(x) + p) = (x(t), du(x(t)) + p(t))$, then we have: $\forall t \in \mathbb{R}, \phi_t^{H_0}(x, du(x) + \frac{1}{\sqrt{C}}p) = (x(\frac{1}{\sqrt{C}}t), du(x(\frac{1}{\sqrt{C}}t)) + \frac{1}{\sqrt{C}}p(\frac{1}{\sqrt{C}}t))$.*

2.3 Proof of theorem 1

In this subsection, we assume that M, $s : M \to M$ and $L : TM \to \mathbb{R}$ satisfy the hypothesis (a), (b), we fix $c > c(L)$ and a strict C^∞ c-subsolution u such that $u \circ s = -u$. Then we know (see appendix B) that: $\forall (x,p) \in T^*M, H(s(x), -p \circ (Ds(x))^{-1}) = H(x,p)$.

Moreover, H_0 is the Hamiltonian function built with H, u and c in the previous subsection and L_0 is the corresponding Lagrangian function . Then, L_0 is s-symmetric.

A loop $\Gamma : [0,T] \to T^*M$ is "s-symmetric" if: $\forall t \in [0,T], \Gamma(t) \circ Ds = -\Gamma(-t)$.

Proposition 12 *If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ satisfy the hypothesis (a), (b), there exists a s-symmetric closed 1-form λ of M such that the Aubry set of the Lagrangian function $L+\lambda$ is the union of some s-symmetric and non critical periodic orbits. Moreover, this result is true for L_0 too.*

Proof Let $\mathcal{F}$ be the set of fixed points of s and B be a connected component of $M\backslash\mathcal{F}$. Then we complete B to a manifold with boundary $\bar{B}$ whose interior is B: if $M\backslash\mathcal{F}$ has two connected components, ∂B is diffeomophic to $\mathcal{F}$ and if $M\backslash\mathcal{F}$ has one connected component, ∂B is diffeomophic to the disjoint union of two copies of $\mathcal{F}$. We know (hypothesis (a)) that ∂B has at least two connected components. We write: $\partial B = \partial B_1 \cup \partial B_2$ where the ∂B_i are disjoint, non empty and compact. Using bump functions, we can construct a C^∞ function $\eta : \bar{B} \to \mathbb{R}$ such that:

- there exists a neighborhood U_1 of $\partial_1 B$ such that $\eta_{|U_1} = 0$;
- there exists a neighborhood U_2 of $\partial_2 B$ such that $\eta_{|U_2} = 1$;

then $d\eta$ is a closed 1-form on $\bar{B}$ which is zero in the neighborhood $U_1 \cup U_2$ of ∂B. We can extend it to a closed 1-form μ on M by: $\forall x \in B, \mu(x) = d\eta(x)$ and: $\forall x \in M\backslash B, \mu(x) = 0$. We define a new 1-form ν: $\nu = \mu - \mu \circ Ds$. Then $\nu \circ Ds = -\nu$ (this means that ν is s-symmetric) and ν is not cohomologically trivial. Indeed, let $\gamma : [0,1] \to \bar{B}$ be a C^1 arc such that $\gamma(0) \in \partial_1 B$, $\gamma(1) \in \partial_2 B$, $\gamma(]0,1[) \subset B$, $\dot\gamma(0)$ is in the eigenspace of $Ds(\gamma(0))$ for the eigenvalue -1 and $\dot\gamma(1)$ is in the eigenspace of $Ds(\gamma(1))$ for the eigenvalue -1. We define a C^1 loop $\tilde\gamma : [0,2] \to M$ by:

- $\forall t \in [0,1], \tilde\gamma(t) = \gamma(t)$;
- $\forall t \in [1,2], \tilde\gamma(t) = s \circ \gamma(2-t)$.

Then by definition: $\int_{\tilde\gamma} \nu = \int_0^1 d\eta(\dot\gamma(t))dt + \int_1^2 (-d\eta \circ Ds(Ds(-\dot\gamma(2-t))))dt = 2\int_0^1 d\eta(\dot\gamma(t))dt = 2(\eta(\gamma(1)) - \eta(\gamma(0))) = 2$. In fact, we have constructed a symmetric closed 1-form η on M which computes the algebraic number of intersections between γ and $\partial_2 B$: such a (non necessarily symmetric) form always exists by the Poincaré duality.

Let us now recall what is a minimizing measure for L satisfying (a) and (b). Let $\mathcal{M}(L)$ be the set of the (f_t^L)-invariant Borel probabilities. Then $\mu_0 \in \mathcal{M}(L)$ is minimizing if: $\forall \mu \in \mathcal{M}(L), \int L d\mu_0 \leq \int L d\mu$. The existence of minimizing measures is proved in [10]. Moreover, it is proved in [1] that the image under the Legendre map of the support of every minimizing measure is contained in the critical level. The closure of the union of the images by the Legendre map of the support of the minimizing measures is called the Mather set and is denoted by $\mathcal{M}(L)$. Let us recall some classical results (see [10] or [6] for the two first items and [7] or [8] for the last item):

1. the Mather set is contained in the Aubry set: $\mathcal{M}(L) \subset \mathcal{A}(L)$;

2. if $p \in \mathcal{A}(L)$, its α and ω-limit sets for the Hamiltonian flow (Φ_t) meet the image under the Legendre map of the support of an ergodic minimizing measure;

3. for every C^1 loop $\Gamma : [0,T] \to T^*M$, if $\gamma = \pi^* \circ \Gamma$ and μ_γ is the probability measure supported by $(\gamma, \dot{\gamma})$, if μ is a minimizing measure: $\int L d\mu \leq \int L d\mu_\gamma$.

Instead of considering the Lagrangian function L, we consider the Lagrangian function $L_t = L + t\nu$ where $t \in \mathbb{R}^*$ and ν is the closed s-symmetric 1-form we constructed before; then it is known that the Lagrangian flow for L_t is equal to the Lagrangian flow for L[4]. We will prove that for a certain t, $\mathcal{M}(L_t)$ is the union of some s-symmetric periodic orbits Γ such that if $\gamma = \pi^* \circ \Gamma$: $\int_\gamma \nu \neq 0$. Let $m = \min L$; if $\tilde{\gamma}$ is the loop which we have built before such that: $\int_{\tilde{\gamma}} \nu = 2$, we select $t \in \mathbb{R}$ such that: $\frac{1}{2}\int_0^2 L(\dot{\tilde{\gamma}}) + t - m < 0$. Let μ be an ergodic minimizing measure for the Lagrangian function L_t and $(x, v) \in TM$ be a generic point for μ; we let: $x(\tau) = \pi \circ f_\tau^L(x, v)$. Then: $\int L_t d\mu \leq \frac{1}{2}\int_0^2 L_t(\dot{\tilde{\gamma}})$ i.e:

$$\lim_{T\to+\infty} \left(\frac{1}{T}\int_0^T (L(\dot{x}(\tau))d\tau + \frac{t}{T}\int_0^T \nu(\dot{x}(\tau))d\tau \right) \leq \frac{1}{2}\int_0^2 L(\dot{\tilde{\gamma}}) + t;$$

This implies:

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T L(\dot{x}(\tau))d\tau + \liminf_{T\to+\infty} \frac{t}{T}\int_0^T \nu(\dot{x}(\tau))d\tau \leq \frac{1}{2}\int_0^2 L(\dot{\tilde{\gamma}}) + t.$$

But we have: $\forall T > 0, \frac{1}{T}\int_0^T (L(\dot{x}(\tau))d\tau \geq m$ then we must have:

$$\liminf_{T\to+\infty} \frac{t}{T}\int_0^T \nu(\dot{x}(\tau))d\tau \leq \frac{1}{2}\int_0^2 L(\dot{\tilde{\gamma}}) + t - m < 0.$$

This implies that the arc $x : \mathbb{R} \to M$ cuts $\partial_2 B \subset \mathcal{F}$ infinitely many times. We may assume for example that $x(0) \in \mathcal{F}$. Let us recall that we know that: $L \circ (-Ds) = L$. Thus $\tau \in \mathbb{R} \to s \circ x(-\tau)$ is a generic orbit for an ergodic minimizing measure too, with the same initial condition $x(0)$ as x . The Aubry set (and thus the Mather set) being a graph, implies

[4]see the appendix C for the links between L and $L + \lambda$ and the associated Hamiltonian functions.

that: $\forall \tau \in \mathbb{R}, s \circ x(-\tau) = x(\tau)$. Doing the same argument for another time $T > 0$ such that $x(T) \in \mathcal{F}$, we obtain: $\forall \tau \in \mathbb{R}, s \circ x(2T - \tau) = x(\tau)$ an thus: $\forall \tau \in \mathbb{R}, x(\tau + 2T) = x(\tau)$.
We have proved that every ergodic minimizing measure for L_t is supported by a periodic orbit which cuts $\mathcal{F}$ infinitely many times and is s-symmetric.

Let $p \in \mathcal{A}(L_t)$. Then its ω-limit set contains one point of the image under the Legendre map of the support of a ergodic minimizing measure; thus if $x : \mathbb{R} \to M$ is the projection of the orbit of p, it cuts $\mathcal{F}$ infinitely many times too. The same argument as before ($\mathcal{A}(L_t)$ is a graph which is s-symmetric) implies that the orbit of p is periodic, s-symmetric and cuts $\mathcal{F}$.

This orbit is not critical because at a critical point we have:

$$\liminf_{T\to+\infty} \frac{t}{T} \int_0^T \nu(\dot{x}(\tau))d\tau = 0$$

an thus we don't have:

$$\liminf_{T\to+\infty} \frac{t}{T} \int_0^T \nu(\dot{x}(\tau))d\tau < 0.$$

This ends the proof for L smooth.

Let us now deal with the case of L_0. The only problem for L_0 is that H_0 is not smooth on the graph of du. The idea is then: we will modify H_0 in a small neighborhood U of the graph of du to obtain a smooth Lagrangian function $\tilde{L}_0$ which satisfy the hypothesis (a), (b) and is such that $\tilde{H}_0$ is equal to H_0 outside U (where $\tilde{H}_0$ is the Hamiltonian function associated to $\tilde{L}_0$). Then if we choose for this $\tilde{L}_0$ a t as before such that $|t| = -t$ is very big, the Aubry set $\mathcal{A}(\tilde{L}_0 + t\nu)$ is contained in $\{(x,p); \tilde{H}_0(x, p - t\nu(x)) = C\}$ with C very big; therefore $\{(x,p); \tilde{H}_0(x, p - t\nu(x)) = C\} = \{(x,p); H_0(x, p - t\nu(x)) = C\}$ and this is the Aubry set for $L_0 + t\nu$ also (because we have seen that the Aubry set depends only on the energy level).
We have now to construct $\tilde{H}_0$; it is easy to smooth a function, the only difficulty is to obtain a strictly convex function. To do that, we used a method explained in [4]:

- at first we consider a positive C^∞ bump function $\eta : \mathbb{R} \to \mathbb{R}_+$ which is zero for $|x| \geq 1$ and such that $\int_{\mathbb{R}} \eta(x^2)dx = 1$;
- then let μ be a s-symmetric Lebesgue measure on T^*M and $\|.\|$ be a Riemannian metric; we define for every $x \in M$ a function $\alpha_x : T^*_x M \to \mathbb{R}_+$ by: $\alpha_x(p) = \eta(\|p\|^2)$; then $k(x) = \int_{T^*_x M} \alpha_x(p) d\mu_x(p)$ is strictly positive and we define: $\alpha(p) = \frac{\alpha_{\pi^*(p)}(p)}{k(\pi^*(p))}$. The function α is then smooth, positive and its integral along any fiber is 1;

- then we define H_ε for $\varepsilon > 0$ by:
$$H_\varepsilon(du(x)+p) = \int_{T_x^*M} \frac{1}{\varepsilon^n}\alpha(\frac{q}{\varepsilon})H_0(du(x)+p+q)d\mu_x(q)$$
$$\text{i.e}\ :\quad H_\varepsilon(du(x)+p) = \int_{T_x^*M} \frac{1}{\varepsilon^n}\alpha(\frac{q-p}{\varepsilon})H_0(du(x)+q)d\mu_x(q);$$
then H_ε is C^∞, s-symmetric, strictly convex (to prove that we notice that in the fiber and when ε tends to 0, H_ε tends uniformly to H_0 on every compact subset. Moreover, on every compact subset where H_0 is smooth, the derivatives on H_ε tend to the derivatives of H_0);

- then we use a C^∞ function $\beta : T^*M \to [0,1]$ which is s-symmetric, zero on a neighborhood V of the zero section and 1 outside a compact neighborhood U of the zero section and we define: $K_\varepsilon(x,p) = (1-\beta(p-du(x)))H_\varepsilon(x,p)+\beta(p-du(x))H_0(x,p)$. For ε sufficiently small, this function is strictly convex because on every compact subset where H_0 is smooth, the derivatives on H_ε tend to the derivatives of H_0; moreover, K_ε is smooth and is equal to H_0 outside U (and equal to H_ε on V).

This ends the proof for L_0. □

Now we consider M, s, L, $c > c(L)$, u, H_0 and L_0 as in the beginning of the subsection. We then use proposition 12 to obtain a closed s-symmetric 1-form λ of M such that the Aubry set of the Lagrangian function $L_0 + \lambda$ is the union of some s-symmetric and non critical periodic orbits for the Hamiltonian function $x \to H_0(x, p - \lambda(x))$. As these orbits are non critical, their energy level is $\{(x,p) \in T^*M; H_0(x, p - \lambda(x)) = C\}$ with $C > 0$. Using proposition 11, we deduce: the Aubry set $\mathcal{A}(L_0 + \frac{1}{\sqrt{C}}\lambda)$ is contained in $\{(x,p) \in T^*M; H_0(x, p - \frac{1}{\sqrt{C}}\lambda(x)) = 1\} = \{(x,p) \in T^*M; H(x, p - \frac{1}{\sqrt{C}}\lambda(x)) = c\}$ and is the union of some s-symmetric and non critical periodic orbits for the Hamiltonian function $x \to H_0(x, p - \frac{1}{\sqrt{C}}\lambda(x))$.
The Hamiltonian function associated to $L_0 + \frac{1}{\sqrt{C}}\lambda$ is h_0 defined by: $h_0(p) = H_0(p - \frac{1}{\sqrt{C}}\lambda)$; if we define: $h(p) = H(p - \frac{1}{\sqrt{C}}\lambda)$ (the associated Lagrangian function is then $\ell = L + \frac{1}{\sqrt{C}}\lambda$), we have then: $\{h_0 = 1\} = \{h = c\}$. As $\{h_0 = 1\}$ is the critical level for $\ell_0 = L_0 + \frac{1}{\sqrt{C}}\lambda$, using proposition 6, we deduce that it is the critical level for $\ell = L + \frac{1}{\sqrt{C}}\lambda$ too. Then, using proposition 7, we deduce that: $\mathcal{A}(L + \frac{1}{\sqrt{C}}\lambda) = \mathcal{A}(L_0 + \frac{1}{\sqrt{C}}\lambda)$.

As $c > c(L)$, we know that there is no critical point for H in $\mathcal{H} = \{H = c\}$ and thus no critical point for h on $\{h = c\}$[5] ; moreover, there is no critical point for H_0 in $\mathcal{H}$ and

[5]The set $\{H \leq c(L)\}$ contains a graph, and therefore: $\forall x \in M, c(L) \geq \min\{H(x,p); p \in T_x^*M\}$.

thus no critical point for h_0 on $\{h_0 = 1\}$ too. Then , there is no critical point for h or h_0 on $\mathcal{K} = \{h_0 = 1\} = \{h = c\}$. Therefore the flow (ϕ_t^h) restricted to $\mathcal{K}$ is a reparametrization of the flow $(\phi_t^{h_0})$ restricted to $\mathcal{K}$. Therefore $\mathcal{A}(L + \frac{1}{\sqrt{C}}\lambda) = \mathcal{A}(L_0 + \frac{1}{\sqrt{C}}\lambda)$ is the union of some s-symmetric and non critical periodic orbits for the Hamiltonian function h and is contained in $\{h = c\}$.
As the Aubry set contains the Mather set, we have finished the proof of theorem 1.
In fact, we have proved a more precise result than theorem 1:

Proposition 13 *If M and $s : M \to M$ satisfy the hypothesis (a), there exists a s-symmetric closed 1-form λ of M such that :*
for every $L : TM \to \mathbb{R}$ satisfying (b) and every $c > c(L)$, there exists $t_c \in \mathbb{R}$ such that the Mather set of the Lagrangian function $L + t_c\lambda$ is contained in $\{(x,p) \in T^*M; H(x, p - t_c\lambda(x)) = c\}$ and is a union of non critical and s-symmetric periodic orbits for the Hamiltonian function associated to the Lagrangian function $L + t_c\lambda$.

3 Proof of theorem 3

If M, $s : M \to M$ and $L : TM \to \mathbb{R}$ satisfy the hypothesis (a), (b) and if $c > c(L)$, we have proved in theorem 1 that there exists a s-symmetric closed 1-form λ of M such that the Mather set of the Lagrangian function $L + \lambda$ is contained in $\{(x,p) \in T^*M; H(x, p - \lambda(x)) = c\}$ and is a union of non critical periodic orbits for the Lagrangian function $L + \lambda$.

3.1 The Green bundles on the Aubry set

We recall in this subsection some well-known results which are proved in [2]. Let (x,p) be a point of T^*M; we call the "vertical" at (x,p) the linear subspace $V(x,p) = \ker D\pi^*(x,v)$ of $T_{(x,p)}(T^*M)$ and we say that the orbit of (x,p) has a conjugate point if there exists $t \neq t'$ such that $D\phi^H_{t-t'}(V(\phi_{t'}(x,p))) \cap V(\phi^H_t(x,p)) \neq \{0\}$.

Proposition 14 *Let (x,p) be a point of the Aubry set of L; then its orbit has no conjugate point.*

In fact, every minimizing orbit (with fixed ends) has no conjugate point.

Proposition 15 *Suppose that the orbit of $(x,p) \in T^*M$ does not contain conjugate points and that $\mathcal{H} = \{H = H(x,p)\}$ is a regular level of H. Then there exists two (ϕ_t^H) invariant Lagrangian subbundles $\mathbb{L}_-, \mathbb{L}_+ \subset T(T^*M)$ along the orbit of (x,p) given by:*

$$\mathbb{L}_-(x,p) = \lim_{t\to+\infty} D\phi^H_{-t}(V(\phi_t(x,p)));$$

$$\mathbb{L}_+(x,p) = \lim_{t\to+\infty} D\phi_t^H(V(\phi_{-t}(x,p))).$$

Moreover, $\mathbb{L}_-(x,p)\cup\mathbb{L}_+(x,p) \subset T_{(x,p)}\mathcal{H}$, $\mathbb{L}_-(x,p)\cap V(x,p) = \mathbb{L}_+(x,p)\cap V(x,p) = \{0\}$, $X_H(x,p)\subset \mathbb{L}_-(x,p)\cap\mathbb{L}_+(x,p)$ *where* X_H *is the Hamiltonian vector field for* H*.*

These subbundles are called the Green bundles along the orbit of (x,p).

Proposition 16 *Suppose that the orbit of* $(x,p)\in T^*M$ *does not contain conjugate points and that* $\mathcal{H} = \{H = H(x,p)\}$ *is a regular level of* H*. Let* $\mathbb{L}_-$ *and* $\mathbb{L}_+$ *be the Green bundles along the orbit of* (x,p)*.*
If (x,p) *is periodic and if* $\mathbb{L}_-$ *and* $\mathbb{L}_+$ *are transverse in* $T\mathcal{H}$ *at every point of the orbit of* (x,p)*, then this orbit is hyperbolic.*

Therefore, to obtain a hyperbolic orbit, we only have to build a periodic orbit in the Aubry set such that, along this periodic orbit, the Green bundles are transverse. We will explain how to do that in the next subsection.

3.2 Construction of transverse Green bundles along a periodic orbit

We need more precise results concerning the Lagrangian subbundles. Let us choose local coordinates $(x^1,\dots,x^n)$ in $U\subset M$ which we complete in dual coordinates of T^*M: the point with coordinates $(x^1,\dots,x^n,p^1,\dots p^n)$ is $\sum_{k=1}^n p^k dx^k$.

If $\mathbb{L}$ is a Lagrangian subspace of $T_{(x,p)}(T^*M)$ which is transverse to the vertical $V(x,p) = \ker D\pi^*(x,p)$, we can read it in these coordinates as the graph of a symmetric matrix. Therefore, if $\mathbb{L}_1$ and $\mathbb{L}_2$ are two such Lagrangian subspaces of $T_{(x,p)}(T^*M)$ which are transverse to the vertical, they are the graphs of two symmetric matrices S_1 and S_2 and we can compute their relative height defined by: $h(\mathbb{L}_1,\mathbb{L}_2) = S_2 - S_1$. It corresponds to $h(\mathbb{L}_1,\mathbb{L}_2)$ a quadratic form defined by: $q(\delta x) = {}^t\delta x(S_2-S_1)\delta x$. The index and the nullity of this quadratic form are in fact independent of the considered chart, and are intrinsically defined as being the index and the nullity of the quadratic form $Q = Q(\mathbb{L}_1,\mathbb{L}_2)$ (ω is the usual symplectic form of T^*M) defined by:

$$\forall u_1\in\mathbb{L}_1, \{u_2\} = \left(D\pi^*(x,p)\right)^{-1}\left(D\pi^*(x,p)u_1\right)\cap\mathbb{L}_2, Q(\mathbb{L}_1,\mathbb{L}_2)(u_1) = Q(u_1) = \omega(u_1,u_2).$$

We say that $\mathbb{L}_2$ is above $\mathbb{L}_1$ if $Q(\mathbb{L}_1,\mathbb{L}_2)$ is positive (i.e. if its index is 0) and that $\mathbb{L}_2$ is strictly above $\mathbb{L}_1$ if $\mathbb{L}_2$ is above $\mathbb{L}_1$ and the dimension of $\mathbb{L}_1\cap\mathbb{L}_2$ is 0 or 1, i.e. if $Q(\mathbb{L}_1,\mathbb{L}_2)$ is positive with nullity 0 or 1.

Concerning the Green bundles of an orbit with no conjugate point, we have:

Proposition 17 *Suppose that the orbit of* $(x,p) \in T^*M$ *does not contain conjugate points and that* $\mathcal{H} = \{H = H(x,p)\}$ *is a regular level of* H*. Let* $\mathbb{L}_-$ *and* $\mathbb{L}_+$ *be the Green bundles along the orbit of* (x,p)*.*
Then $\mathbb{L}_+$ *is above* $\mathbb{L}_-$*. Moreover, if* (x,p) *is periodic and* $\mathbb{L}_+$ *is strictly above* $\mathbb{L}_-$ *at every point of the orbit of* (x,p)*, then this orbit is hyperbolic.*

Proof The first assertion is very classical, and is a consequence of the construction of $\mathbb{L}_+$ and $\mathbb{L}_-$; more precisely we have by definition:

$$\mathbb{L}_-(x,p) = \lim_{t\to+\infty} D\phi_{-t}(V(\phi_t(x,p)));$$

$$\mathbb{L}_+(x,p) = \lim_{t\to+\infty} D\phi_t(V(\phi_{-t}(x,p))).$$

As H is strictly convex, for $t > 0$ small enough, for every point (x',p') of the orbit of (x,p) (every orbit is relatively compact), $Q(D\phi_{-t}(V(\phi_t(x',p'))), D\phi_t(V(\phi_{-t}(x',p'))))$ is definite positive. Let us prove that: we choose a finite atlas of M and then we use in T^*M the dual coordinates as described before. These coordinates are symplectic, and in these coordinates the matrix of $D\phi_t(x',p')$ is:

$$M_t \quad = \quad \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}$$

which is a symplectic matrix (of course it depends on the considered chart used). Using the Hamilton equations, we obtain that: $c^t \sim_{t\to 0} tH_{pp}(x',p')$ and $d^t \sim_{t\to 0} \mathbf{1}$ and $a^t \sim_{t\to 0} \mathbf{1}$, this being uniform because the orbit is relatively compact and we have a finite number of charts. But in these coordinates, if $t \neq 0$ is small enough, $D\phi_t(x',p')(V(x',p'))$ is the graph of the symmetric matrix $d^t\,(c^t)^{-1} \sim_{t\to 0} \frac{1}{t}\,(H_{pp}(x',p'))^{-1}$ which is positive definite for $t > 0$, negative definite for $t < 0$. This implies that for $t > 0$ small enough, $Q(D\phi_{-t}(V(\phi_t(x',p'))), D\phi_t(V(\phi_{-t}(x',p'))))$ is definite positive.

But as the orbit has no conjugate point, for every $t > 0$, the nullity of $Q(D\phi_{-t}(V(\phi_t(x,p))), D\phi_t(V(\phi_{-t}(x,p))))$ is null, therefore its index is constant ;

$$\forall t > 0, Q(D\phi_{-t}(V(\phi_t(x,p))), D\phi_t(V(\phi_{-t}(x,p)))) \geq 0$$

When we take the limit, we obtain: $Q(\mathbb{L}_+, \mathbb{L}_-) \geq 0$ i.e. $\mathbb{L}_+$ is above $\mathbb{L}_-$.

The second assertion is a corollary of proposition 16.

□

Proposition 18 *Let $s : M \to M$ satisfy (a) and let $L_0 : TM \to \mathbb{R}$ satisfying (b); let $q_1, q_2 \in \mathbb{Q}$ such that $0 < q_1 < q_2$. The following property is open and dense in the set of the Lagrangian functions of the form: $L = L_0 + \varphi$ where $\varphi \in C_s^\infty(M)$:*
"there exists a s-symmetric closed 1-form λ on M such that $\mathcal{A}(L+\lambda)$ is a symmetric periodic hyperbolic orbit and such that $\mathcal{A}(L+\lambda) \subset \{(x,p) \in T^*M; H(x, p-\lambda(x)) \in]c(L)+q_1, c(L)+q_2[\}$".

Proof Let us prove that this property is open.
Let $L = L_0 + \varphi_0$ satisfy this property. Then there exists a symmetric closed 1-form λ such that $\mathcal{A}(L+\lambda)$ is one symmetric periodic hyperbolic orbit and such that $\mathcal{A}(L+\lambda) \subset \{(x,p) \in T^*M; H(x, p-\lambda(x)) = c\}$ with $c(L+\lambda) = c \in]c(L)+q_1, c(L)+q_2[$. One characterization of $c(L)$ is (see [7] or [8]): $-c(L)$ is the infimum of the $\int L d\mu$ where μ is in the set of the probability measures of TM supported by a C^1 periodic loop of M. This implies that $c(L+\varphi)$ depends continuously on φ. Therefore there exists an open subset $\mathcal{U}_0$ of $C_s^\infty(M)$ containing 0 such that for every $\varphi \in \mathcal{U}_0$, $c(L+\lambda+\varphi) \in]c(L+\varphi)+q_1, c(L+\varphi)+q_2[$. Therefore for $\varphi \in \mathcal{U}_0$, we have: $\mathcal{A}(L+\lambda+\varphi) \subset \{(x,p) \in T^*M; H(x, p-\lambda(x)) - \varphi(x) \in]c(L+\varphi)+q_1, c(L+\varphi)+q_2[\}$[6].
To go on with the proof of the openness of the property, we need now to introduce another well-known set, the so called Manẽ set $\mathcal{N}(L)$ of L. In fact, the reader doesn't need to know its definition, but only some of its properties, which are proved in [5]:

1. the Mañé set $\mathcal{N}(L+\varphi)$ depends upper semi-continuously on φ;
2. $\mathcal{A}(L) \subset \mathcal{N}(L)$;
3. if L has only one minimizing measure, then $\mathcal{N}(L) = \mathcal{A}(L)$.

Now, as $\mathcal{A}(L+\lambda)$ is one periodic orbit, then $L+\lambda$ has only one minimizing measure; therefore $\mathcal{N}(L+\lambda) = \mathcal{A}(L+\lambda)$; as the Mañé set depends upper semi-continuously on φ and contains the Aubry set, the function $\varphi \to \mathcal{A}(L+\lambda+\varphi)$ is upper semi-continuous at 0. But for $\varphi = 0$, this set is a hyperbolic periodic orbit Γ_0; therefore, for φ small enough ($\varphi \in \mathcal{U}_1$), and c close enough to $c(L+\lambda)$, $L+\lambda+\varphi$ has a closed hyperbolic orbit very close to Γ_0 contained in $\{(x,p); H(x, p+\lambda(x)) - \varphi(x) = c\}$ and there is no other orbit which stays close to this orbit in this Hamiltonian level. Then necessarily, $\mathcal{A}(L+\varphi)$ is one hyperbolic periodic orbit Γ, which is close to Γ_0; therefore it cuts $\mathcal{F}$ and therefore it's s-symmetric.

Now we have to prove that the property is dense. Let $L = L_0 + \varphi_0$ and $c \in]c(L) + q_1, c(L) + q_2[$. Then we know (see theorem 1) that there exists a s-symmetric closed 1-form

[6] We recall that the Hamiltonian function associated to $L+\varphi$ is $H - \varphi$ (see appendix C).

λ of M such that the Mather set of the Lagrangian function $L+\lambda$ is contained in $\{(x,p)\in T^*M; H(x,p-\lambda(x))=c\}$ and is a union of non critical and s-symmetric periodic orbits for the Hamiltonian function associated to the Lagrangian function $L+\lambda$. Moreover, as the Mather set is a graph above a compact part of the zero section, if we choose $(x,p)\in\mathcal{A}(L+\lambda)$, it is a periodic point for the Hamiltonian flow of $(x,p)\to H(x,p-\lambda(x))$ and its orbit Γ is a graph above $\gamma=\pi^*\circ\Gamma$. Now let $\psi: M\to\mathbb{R}$ be a C^∞ function which is s-symmetric, null on γ and strictly positive on $M\backslash\gamma$. Let us recall a characterization (see [7]) of the minimizing measures :
we define $\mathcal{M}$ as being the set of Borel probability measures μ on TM such that $\int\|v\|d\mu<\infty$ and such that for every $g\in C^\infty(M)$, $\int dg(x)vd\mu=0$; these measures are named "holonomic"; then μ_0 is minimizing for L if and only if $\mu_0\in\mathcal{M}$ and for every $\mu\in\mathcal{M}$, $\int Ld\mu_0\leq\int Ld\mu$.
Let us use this characterization to prove that $L+\lambda+\psi$ has a unique minimizing measure, the one supported by Γ. At first, let us notice that Γ is a periodic orbit for $L+\lambda+\psi$ because along Γ, the partial derivatives of $L+\lambda+\psi$ are the same as the ones of $L+\lambda$. Then we name μ_0 the probability measure supported by Γ, and we consider $\mu\in\mathcal{M}\backslash\{\mu_0\}$. There are two cases:

1. μ is not minimizing for $L+\lambda$; then $\int(L+\lambda+\psi)d\mu\geq\int(L+\lambda)d\mu>\int(L+\lambda)d\mu_0=\int(L+\lambda+\psi)d\mu_0$; therefore μ is not minimizing for $L+\lambda+\psi$;
2. μ is minimizing for $L+\lambda$; therefore it is the measure supported by a periodic orbit Γ_1 and we have, the Mather set being a graph: $\pi^*(\Gamma)\cap\pi^*(\Gamma_1)=\emptyset$; then there exists $m>0$ such that: $\forall t\in\mathbb{R}, \psi(\pi^*(\Gamma_1(t)))\geq m$ and then: $\int(L+\lambda+\psi)d\mu=\int(L+\lambda)d\mu_0+\int\psi d\mu\geq m+\int(L+\lambda)d\mu_0>\int(L+\lambda)d\mu_0=\int(L+\lambda+\psi)d\mu_0$.

Therefore μ_0 is the unique minimizing measure of $L+\lambda+\psi$.
We want now to add another function ψ_1 to $\tilde{L}=L+\psi$ which is s-symmetric, zero on γ and positive on $M\backslash\gamma$ such that for $\tilde{L}+\lambda+\psi_1=L+\lambda+\psi+\psi_1$: if $\mathbb{L}'_+$ and $\mathbb{L}'_-$ are the Green bundles for $\tilde{L}+\lambda+\psi_1$ along Γ, then $\mathbb{L}'_+$ is strictly above $\mathbb{L}'_-$. By proposition 17, this will imply that Γ is hyperbolic for $\tilde{L}+\lambda+\psi_1$ and will conclude the proof.
Let t_0 be such that $x_0=\gamma(t_0)\notin\mathcal{F}$. We choose C^∞-coordinates $(x^1,\dots,x^n)$ in a neighborhood $U\subset M\backslash\mathcal{F}$ of x_0 such that if $U\cap\gamma=\{\gamma(t); t\in]t_0-\varepsilon,t_0+\varepsilon[\}$, then: $\forall t\in]t_0-\varepsilon,t_0+\varepsilon[, (x^1,\dots,x^n)(\gamma(t))=(t,0\dots,0)$. We work then in the dual (symplectic) coordinates $(x^1,\dots,x^n,p^1,\dots,p^n)$ on T^*U. We define a function $\psi_1: M\to\mathbb{R}$ which is:

- zero on $M\backslash U$;
- defined in the chart U by: $\psi_1(x)=\eta\left(\sum_{i=1}^n\left(x^i\right)^2\right)\sum_{j=2}^n\left(x^j\right)^2$ where $\eta:\mathbb{R}\to[0,1]$ is a C^∞ function which is zero outside $]-\left(\frac{\varepsilon}{2}\right)^2,\left(\frac{\varepsilon}{2}\right)^2[$ and strictly positive in $]-\left(\frac{\varepsilon}{2}\right)^2,\left(\frac{\varepsilon}{2}\right)^2[$.

The function ψ_1 is not s-symmetric for the moment, but we will soon modify it to obtain a s-symmetric function.
Then the Hamiltonian function associated to $\tilde{L}+\psi_1$ is $\tilde{H}-\psi_1$ (where $\tilde{H}$ is the Hamiltonian function associated to $\tilde{L}$) and Γ is a periodic orbit for the Hamiltonian vector fields of $\tilde{H}$ and $\tilde{H}-\psi_1$ and is in the Aubry set for $\tilde{L}$ and $\tilde{L}+\psi_1$. We call T the period of γ.
Let $\mathbb{L}_+$ and $\mathbb{L}_-$ be the Green bundles for $\tilde{L}$ along Γ; we know that $\mathbb{L}_+$ is above $\mathbb{L}_-$. We will prove:

Lemma 19 *Let us introduce the notations:*

- *for* $t\in\mathbb{R}$ *and* $(x,p)\in\Gamma$, $\mathbb{L}_+^t(x,p)=D\phi_t^{\tilde{H}-\psi_1}(\mathbb{L}_+(\phi_{-t}^{\tilde{H}}(x,p));$
- *for* $t\in\mathbb{R}$ *and* $(x,p)\in\Gamma$, $\mathbb{L}_-^t(x,p)=D\phi_{-t}^{\tilde{H}-\psi_1}(\mathbb{L}_-(\phi_t^{\tilde{H}}(x,p)).$

Then for every $t>0$*, we have:*

1. $\mathbb{L}_+^t(x,p)$ *is transverse to* $V(x,p)$ *and* $D\phi_t^{\tilde{H}-\psi_1}(V(\phi_{-t}^{\tilde{H}}(x,p)))$ *is strictly above* $\mathbb{L}_+^t(x,p)$*;*
2. $\mathbb{L}_-^t(x,p)$ *is transverse to* $V(x,p)$ *and* $\mathbb{L}_-^t(x,p)$ *is strictly above* $D\phi_{-t}^{\tilde{H}-\psi_1}(V(\phi_t^{\tilde{H}}(x,p)))$*;*
3. $\mathbb{L}_+^t(x,p)$ *is above* $\mathbb{L}_+(x,p)$ *(strictly above if* $t\geq T$*);*
4. $\mathbb{L}_-(x,p)$ *is above* $\mathbb{L}_-^t(x,p)$ *(strictly above if* $t\geq T$*).*

Let us explain how this lemma allows us to conclude. The notations $\mathbb{L}'_+$, $\mathbb{L}'_-$ will designate the Green bundles for $\tilde{L}+\psi_1$ along Γ and we want to prove that $\mathbb{L}'_+$ is strictly above $\mathbb{L}'_-$. We will prove that $\mathbb{L}'_+$ is strictly above $\mathbb{L}_+$ and that $\mathbb{L}_-$ is strictly above $\mathbb{L}'_-$; as $\mathbb{L}_+$ is above $\mathbb{L}_-$, this will give the conclusion. We will only prove that $\mathbb{L}'_+$ is strictly above $\mathbb{L}_+$, the proof of the result for $\mathbb{L}_-$ follows the same arguments.
We know that: $\mathbb{L}'_+(x,p)=\lim\limits_{n\to+\infty}D\phi_{nT}^{\tilde{H}-\psi_1}(V(x,p))$. Moreover:

a) by 1: for every $t>0$ and every $(x,p)\in\Gamma$, $D\phi_t^{\tilde{H}-\psi_1}(V(\phi_{-t}^{\tilde{H}}(x,p)))$ is strictly above $\mathbb{L}_+^t(x,p)$;

b) by 3, for every $(x,p)\in\Gamma$, $D\phi_T^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))$ is strictly above $\mathbb{L}_+(\phi_T^{\tilde{H}}(x,p))$. This means that $D\phi_T^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))\cap\mathbb{L}_+(\phi_T^{\tilde{H}}(x,p))=\mathbb{R}X_{\tilde{H}}(\phi_T^{\tilde{H}}(x,p))$ where $X_{\tilde{H}}$ is the Hamiltonian vector field. Therefore, for every $u\geq 0$,$D\phi_{T+u}^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))\cap D\phi_u^{\tilde{H}-\psi_1}(\mathbb{L}_+(\phi_T^{\tilde{H}}(x,p))=\mathbb{R}X_{\tilde{H}}(\phi_{T+u}^{\tilde{H}}(x,p))$. Then the nullity of the relative height between $D\phi_{T+u}^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))$ and $D\phi_u^{\tilde{H}-\psi_1}(\mathbb{L}_+(\phi_T^{\tilde{H}}(x,p))$ is constant with its kernel depending continuously on u, and therefore its index is constant equal to zero. For $u = (n-1)T, \dots, T$ we find: $D\phi_{nT}^{\tilde H-\psi_1}(\mathbb{L}_+(x,p))$ is strictly above $D\phi_{(n-1)T}^{\tilde H-\psi_1}(\mathbb{L}_+(x,p))$ which is strictly above $D\phi_{(n-2)T}^{\tilde H-\psi_1}(\mathbb{L}_+(x,p))\dots$ which is strictly above $D\phi_{2T}^{\tilde H-\psi_1}(\mathbb{L}_+(x,p))$ which is strictly above $D\phi_{T}^{\tilde H-\psi_1}(\mathbb{L}_+(x,p))$. Finally, $\mathbb{L}_+^{nT}(x,p)$ is strictly above $\mathbb{L}_+^{T}(x,p)$.

c) using a) for $t = nT$ and b) we obtain: $D\phi_{nT}^{\tilde H-\psi_1}(V(x,p))$[7] is strictly above $\mathbb{L}_+^{nT}(x,p)$ which is above $\mathbb{L}_+^{T}(x,p)$ which is strictly above $\mathbb{L}_+(x,p)$.

We deduce that $\mathbb{L}_+'(x,p) = \lim\limits_{n\to+\infty} D\phi_{nT}^{\tilde H-\psi_1}(V(x,p))$ is above $\mathbb{L}_+^{T}(x,p)$, which is strictly above $\mathbb{L}_+(x,p)$, therefore $\mathbb{L}_+'(x,p)$ is strictly above $\mathbb{L}_+(x,p)$. This finishes the proof of proposition 18.

We have now to prove lemma 19.
We work in some dual charts as explained at the beginning of this subsection, this chart being U near $\Gamma(t_0)$. In such a chart, the coordinates of $\phi_t^{\tilde H}(x,p)$ are $(x_0(t), p_0(t))$ and the coordinates of $\phi_t^{\tilde H-\psi_1}(x,p)$ are $(x_1(t), p_1(t))$. Therefore, the Hamilton equations for $\tilde H - \psi_1$ are:

$$\dot x_1 = \frac{\partial \tilde H}{\partial p}(x,p); \quad \dot p_1 = -\frac{\partial \tilde H}{\partial x}(x,p) + \frac{\partial \psi_1}{\partial x}(x).$$

We will say that $(\delta x, \delta p) : \mathbb{R} \to T(T^*M)$ is an infinitesimal solution along the orbit of (x,p) for (ϕ_t) if $(\delta x(t), \delta p(t)) \in T_{\phi_t(x,p)}(T^*M)$ and $(\delta x(t), \delta p(t)) = D\phi_t(\delta x(0), \delta p(0))$. Let $(\delta x_1, \delta p_1)$ (resp. $(\delta x_0, \delta p_0)$) be an infinitesimal solution for $\tilde H - \psi_1$ (resp. $\tilde H$) along Γ. They satisfy the so-called linearized Hamilton equations (given in coordinates):

$$\delta \dot x_1(t) = \frac{\partial^2 \tilde H}{\partial x \partial p}\delta x_1 + \frac{\partial^2 \tilde H}{\partial p^2}\delta p_1; \quad \delta \dot p_1(t) = -\frac{\partial^2 \tilde H}{\partial x^2}\delta x_1 - \frac{\partial^2 \tilde H}{\partial p \partial x}\delta p_1 + \frac{\partial^2 \psi_1}{\partial x^2}(x)\delta x_1;$$

$$\delta \dot x_0(t) = \frac{\partial^2 \tilde H}{\partial x \partial p}\delta x_0 + \frac{\partial^2 \tilde H}{\partial p^2}\delta p_0; \quad \delta \dot p_0(t) = -\frac{\partial^2 \tilde H}{\partial x^2}\delta x_0 - \frac{\partial^2 \tilde H}{\partial p \partial x}\delta p_0.$$

We are interested in some infinitesimal solutions having the same initial values: $(\delta x_0(0), \delta p_0(0)) = (\delta x_1(0), \delta p_1(0))$. We deduce from the linearized Hamilton equations that, uniformly for $(x,p) \in \Gamma$, if the two infinitesimal solutions have the same initial values, for t close to 0:

$$(*) \quad \delta x_1(t) = \delta x_0(t) + O(t^2); \quad \delta p_1(t) = \delta p_0(t) + t\frac{\partial^2 \psi_1}{\partial x^2}(x)\delta x_1(t) + O(t^2).$$

[7] Let us notice that $\phi_{-nT}^{\tilde H-\psi_1}(x,p) = (x,p)$.

As we are in a dual chart, we may say that $\mathbb{L}_+$ is the graph of a symmetric matrix S_+. Then we have: $\forall t, \delta p_0(t) = S_+(x(t), p(t))\delta x_0(t)$. Let us now prove the results 1. and 3. of lemma 19 for $\mathbb{L}_+$ for $t > 0$ small enough.

We begin by proving that for $t > 0$ small enough, $\mathbb{L}_+^t(\phi_t^{\tilde{H}-\psi_1}(x,p)) = D\phi_t^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))$ is above $\mathbb{L}_+(\phi_t^{\tilde{H}-\psi_1}(x,p))$ and even strictly above for $(x,p) = \Gamma(t_0)$.
As $\mathbb{L}_+(x,p)$ is transverse to $V(x,p)$, for t small enough ($|t| \leq \delta$), $D\phi_t^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))$ is transverse to $V(\phi_t^{\tilde{H}-\psi_1}(x,p))$ and thus we may compute the relative height between $D\phi_t^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))$ and $\mathbb{L}_+(\phi_t^{\tilde{H}-\psi_1}(x,p))$. As $D\phi_t^{\tilde{H}-\psi_1}(\mathbb{L}_+(x,p))$ is Lagrangian and transverse to the vertical, it is the graph of a symmetric matrix $S_+^t(\phi_t^{\tilde{H}}(x,p))$. If $(\delta x_0, \delta p_0)$ and $(\delta x_1, \delta p_1)$ designate infinitesimal solutions along Γ for $(\phi_t^{\tilde{H}})$ and $(\phi_t^{\tilde{H}-\psi_1})$ with the same initial condition $(\delta x_0(0), \delta p_0(0)) = (\delta x_1(0), \delta p_1(0)) \in \mathbb{L}_+(x,p)$, we have then:

$$\delta p_0(t) = S_+(x(t), p(t))\delta x_0(t) \quad \text{and} \quad \delta p_1(t) = S_+^t(x(t), p(t))\delta x_1(t).$$

When $t \in]0, \delta]$ and $(\delta x_0, \delta p_0)$ and $(\delta x_1, \delta p_1)$ have the same initial conditions for $-t$, we deduce:

$$\delta p_0(0) = S_+(x,p)\delta x_0(0) \quad \text{and} \quad \delta p_1(0) = S_+^t(x,p)\delta x_1(0),$$

with $(*)$, this implies:

$$\delta p_0(0) = S_+(x,p)\delta x_0(0) \quad \text{and} \quad S_+^t(x,p)\delta x_0(0) = \delta p_0(0) + t\frac{\partial^2 \psi_1}{\partial x^2}(x)\delta x_0(0) + O(t^2).$$

But along γ, we have: $\frac{\partial^2 \psi_1}{\partial x^2} \geq 0$. We deduce that: $\frac{\partial S_+^t}{\partial t}(x,p)_{|t=0} = \frac{\partial^2 \psi_1}{\partial x^2}(x) \geq 0$. Moreover, if $t \in]t_0 - \frac{\varepsilon}{2}, t_0 + \frac{\varepsilon}{2}[$ then the nullity of $\frac{\partial^2 \psi_1}{\partial x^2}(x(t))$ is 1 and its kernel is $\dot{\gamma}(t)$. We deduce that for every $[-\varepsilon_1, \varepsilon_1] \subset] - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$, there exists $t_1 = t_1(\varepsilon_1) \in]0, \delta]$ such that:

$$\forall t \in [t_0 - \varepsilon_1, t_0 + \varepsilon_1], \forall u \in]0, t_1], \mathbb{L}_+^u(x(t), p(t)) \quad \text{is strictly above} \quad \mathbb{L}_+(x(t), p(t)).$$

Now let $\varepsilon_2 > \varepsilon_1$ be such that $[-\varepsilon_2, \varepsilon_2] \subset] - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. As before, we can associate to ε_2 a $t_2 = t_1(\varepsilon_2)$ and we may assume that $t_2 < t_1$. We consider $u \in]0, t_1]$ and we choose $n \geq 1$ such that $\tau = \frac{u}{n} < \min\{t_2, \varepsilon_2 - \varepsilon_1\}$. Then for $(x', p') \in \Gamma([t_0 - \varepsilon_2, t_0 + \varepsilon_2])$, we have: $\mathbb{L}_+^\tau(x', p') = D\phi_\tau^{\tilde{H}-\psi_1}(\mathbb{L}_+(\phi_{-\tau}(x', p')))$ is strictly above $\mathbb{L}_+(x', p')$. Therefore (we used a similar argument to the previous one: the intersection is 1-dimensional) for every $u \in [0, \delta]$, $D\phi_u^{\tilde{H}-\psi_1}(\mathbb{L}_+^\tau(x', p'))$ is strictly above $D\phi_u(\mathbb{L}_+(x', p'))$. We deduce that if $(x', p') \in \Gamma([t_0 - \varepsilon_2 + u, t_0 + \varepsilon_2])$, we have successively that: $\mathbb{L}_+^u(x', p') = D\phi_u^{\tilde{H}-\psi_1}(\mathbb{L}_+(\phi_{-u}^{\tilde{H}-\psi_1}(x', p'))) = D\phi_{n\tau}^{\tilde{H}-\psi_1}(\mathbb{L}_+(\phi_{-n\tau}^{\tilde{H}-\psi_1}(x', p'))) = D\phi_{(n-1)\tau}^{\tilde{H}-\psi_1}\mathbb{L}_+^\tau(\phi_{-(n-1)\tau}^{\tilde{H}}(x', p'))$ is strictly above $D\phi_{(n-1)\tau}^{\tilde{H}-\psi_1}\mathbb{L}_+(\phi_{-(n-1)\tau}^{\tilde{H}}(x', p'))$ which is strictly above $D\phi^{\tilde{H}-\psi_1}_{(n-2)\tau}\mathbb{L}_+(\phi^{\tilde{H}}_{-(n-2)\tau}(x',p'))$ ... which is strictly above $D\phi^{\tilde{H}-\psi_1}_{\tau}\mathbb{L}_+(\phi^{\tilde{H}}_{-\tau}(x',p'))$ which is strictly above $\mathbb{L}_+(x',p')$ and finally $\mathbb{L}^u_+(x',p')$ is strictly above $\mathbb{L}_+(x',p')$. We obtain:

$$\forall u \in]0,t_1], \forall (x',p') \in \Gamma(]t_0-\frac{\varepsilon}{2}+u, t_0+\frac{\varepsilon}{2}[), \mathbb{L}^u_+(x',p') \quad \text{is strictly above} \quad \mathbb{L}_+(x',p').$$

Let us notice that if (x',p') is a point of Γ which is not in $\Gamma(]t_0-\frac{\varepsilon}{2}, t_0+\frac{\varepsilon}{2}[)$, then if $t_- = \max\{t \leq 0, \phi_t(x',p') \in \Gamma([t_0-\frac{\varepsilon}{2}, t_0+\frac{\varepsilon}{2}]\}$, for every $t \in]0,-t_-]$, $\mathbb{L}^t_+(x',p') = \mathbb{L}_+(x',p')$.
Therefore, if $(x',p') \in \Gamma$, we have different cases:

- if $(x',p') \in \Gamma(]t_0-\frac{\varepsilon}{2}+t_1, t_0+\frac{\varepsilon}{2}[)$ we have proved that $\mathbb{L}^{t_1}_+(x',p')$ is strictly above $\mathbb{L}_+(x',p')$;
- therefore by continuity, for $(x',p') \in \{\Gamma(t_0-\frac{\varepsilon}{2}+t_1), \Gamma(t_0+\frac{\varepsilon}{2})\}$, $\mathbb{L}^{t_1}_+(x',p')$ is above $\mathbb{L}_+(x',p')$;
- if $(x',p') \in \Gamma([0,T]\backslash[t_0-\frac{\varepsilon}{2}, t_0+\frac{\varepsilon}{2}+t_1])$, then $\mathbb{L}^{t_1}_+(x',p') = \mathbb{L}_+(x',p')$;
- if $(x',p') = \Gamma(t_0-\tau) \in \Gamma(]t_0-\frac{\varepsilon}{2}, t_0-\frac{\varepsilon}{2}+t_1])$, then $\mathbb{L}^{t_1}_+(x',p') = \mathbb{L}^{\frac{\varepsilon}{2}-\tau}_+(x',p')$ is above $\mathbb{L}_+(x',p')$;
- the case $(x',p') \in \Gamma(]t_0+\frac{\varepsilon}{2}, t_0+\frac{\varepsilon}{2}+t_1[)$ is similar.

Finally we have found $t_1 > 0$ such that for every $u \in]0,t_1]$ and every $(x',p') \in \Gamma$, $\mathbb{L}^u_+(x',p')$ is above $\mathbb{L}_+(x',p')$, and strictly above at $\Gamma(t_0)$.
Therefore there exists $t_3 > 0$ such that for every $(x,p) \in \Gamma$ and every $t \in]0,t_3]$, $D\phi^{\tilde{H}-\psi_1}_t(V(\phi_{-t}(x,p)))$ is strictly above $\mathbb{L}^t_+(x,p)$ which is above $\mathbb{L}_+(x,p)$ (strictly if $(x,p) = \Gamma(t_0)$).

Now we want to obtain a result for "big" times t. In order to do that, let us prove

Lemma 20 *We consider $(x,p) \in \Gamma$, $\tau > 0$ and $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$ three Lagrangian subspaces of $T_{(x,p)}(T^*M)$ transverse to $V(x,p)$ such that:*

a) $\mathbb{L}_1$ is above $\mathbb{L}_2$ which is above $\mathbb{L}_3$ (resp. strictly above $\mathbb{L}_3$);

b) for every $t \in [0,\tau]$, $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_1)$ and $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_3)$ are transverse to $V(\phi^{\tilde{H}}(x,p))$.

Then for every $t \in [0,\tau]$, $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_2)$ is transverse to $V(\phi^{\tilde{H}}(x,p))$ and $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_1)$ is above $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_2)$ which is above $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_3)$ (resp. strictly above $D\phi^{\tilde{H}-\psi_1}_t(\mathbb{L}_3)$).

Proof We define τ_0 as being the upper bound of the $\tau_1 \in [0,\tau]$ such that for every $t \in [0,\tau_1]$, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ is transverse to $V(\phi_t^{\tilde H}(x,p))$ and $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_1)$ is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ which is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_3)$. Then this upper bound is a maximum: indeed, if $D\phi_{\tau_0}^{\tilde H-\psi_1}(\mathbb{L}_2)$ is not transverse to $V(\phi_{\tau_0}^{\tilde H}(x,p))$, for $t<\tau_0$ close to τ_0, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ contains a vector which is very close to the vertical (in the projective sense); if we choose coordinates near $\phi_{\tau_0}^{\tilde H}(x,p)$, for $t<\tau_0$ close to τ_0, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ is the graph of a symmetric matrix whose norm is very big; this matrix being symmetric, this implies that one of its eigenvalues λ is such that $|\lambda|$ is very big, and if v is a corresponding eigenvector, we have: $\lambda\|v\|^2 < S_3^t(v,v)$ or $S_1^t(v,v)<\lambda\|v\|^2$ (where $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_i)$ is the graph of S_i^t); this contradict that $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_1)$ is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ which is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_3)$. Therefore $D\phi_{\tau_0}^{\tilde H-\psi_1}(\mathbb{L}_2)$ is transverse to $V(\phi_{\tau_0}^{\tilde H}(x,p))$ and S_2^t is defined even for $t=\tau_0$. As the considered relative heights depend continuously on t, we obtain that for $t=\tau_0$ too we have: $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_1)$ is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ which is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_3)$.
Let us now prove that $\tau_0=\tau$. Let us assume that $\tau_0<\tau$. As $D\phi_{\tau_0}^{\tilde H-\psi_1}(\mathbb{L}_2)$ is transverse to $V(\phi_{\tau_0}^{\tilde H}(x,p))$, if $t>\tau_0$ is close to τ_0, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ is transverse to $V(\phi_t^{\tilde H}(x,p))$. Therefore for $t>\tau_0$ close to τ_0, we can deal with the relative heights of the $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_i)$. If (i,j) is fixed, the dimension of $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_i)\cap D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_j)$ is constant when t varies, i.e. the kernel of $S_i^t-S_j^t$ depends continuously on t, therefore its orthogonal too and thus the index of $S_i^t-S_j^t$ is constant. This proves that for $t>\tau_0$ close to τ_0, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_1)$ is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ which is above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_3)$. This contradicts the definition of τ_0, then $\tau_0=\tau$.
Moreover, if we assume that $\mathbb{L}_2$ is strictly above $\mathbb{L}_3$, the same arguments (the index is constant) proves that for every $t\in[0,\tau]$, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_2)$ is strictly above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_3)$. □

Let $(x_1,p_1)\in\Gamma$. We choose $\mathbb{L}_1(x_1,p_1)=D\phi_\delta^{\tilde H-\psi_1}(V(\phi_{-\delta}^{\tilde H-\psi_1}(x_1,p_1)))$ for a certain $\delta>0$. Then we know that for every $t>0$, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_1)$ is transverse to the vertical because there is no conjugate point along Γ. Moreover, we have found $t_3>0$ such that for every $(x,p)\in\Gamma$ and every $t\in]0,t_3]$, $D\phi_t^{\tilde H-\psi_1}(V(\phi_{-t}(x,p)))$ is strictly above $\mathbb{L}_+^t(x,p)$ which is above $\mathbb{L}_+(x,p)$ (strictly if $(x,p)=\Gamma(t_0)$). We define then: $\mathbb{L}_2(x_1,p_1)=D\phi_\delta^{\tilde H-\psi_1}(\mathbb{L}_+(\phi_{-\delta}(x_1,p_1)))=\mathbb{L}_+^\delta(x_1,p_1)$ and $\mathbb{L}_3(x_1,p_1)=\mathbb{L}_+(x_1,p_1)$; therefore if $\delta\in]0,t_3]$, we have: $\mathbb{L}_1(x_1,p_1)$ is strictly above $\mathbb{L}_2(x_1,p_1)$ which is above $\mathbb{L}_3(x_1,p_1)$. Then for every $t\in[0,t_3]$, by the definition of t_3, $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_1(x_1,p_1))$ is strictly above $D\phi_t^{\tilde H-\psi_1}(\mathbb{L}_3(x_1,p_1))$ which is above $\mathbb{L}_3(\phi_t^{\tilde H}(x,p))$. But by definition of t_3, for every $u\in[0,t_3]$, $D\phi_{u+t}^{\tilde H-\psi_1}(\mathbb{L}_1(x_1,p_1))$ is transverse to the vertical and $D\phi_u^{\tilde H-\psi_1}(\mathbb{L}_3(\phi_t^{\tilde H}(x,p)))$ is transverse to the vertical and above $\mathbb{L}_3(\phi_{u+t}^{\tilde H}(x,p))$; using lemma 20, we deduce that for such a u: $D\phi_{u+t}^{\tilde H-\psi_1}(\mathbb{L}_1(x_1,p_1))$ is strictly above $D\phi_{u+t}^{\tilde H-\psi_1}(\mathbb{L}_3(x_1,p_1))$ which is above $D\phi_u^{\tilde{H}-\psi_1}(\mathbb{L}_3(\phi_t^{\tilde{H}}(x,p)))$ which is above $\mathbb{L}_3(\phi_{u+t}^{\tilde{H}}(x,p))$. Repeating this argument, we obtain that for every n-uple $(u_1,\dots,u_n)\in[0,t_1]^n$, then $D\phi_{u_1+\cdots+u_n}^{\tilde{H}-\psi_1}(\mathbb{L}_1(x_1,p_1))$ is strictly above $D\phi_{u_1+\cdots+u_n}^{\tilde{H}-\psi_1}(\mathbb{L}_3(x_1,p_1))$ which is above $\mathbb{L}_3(\phi_{u_1+\cdots+u_n}^{\tilde{H}}(x,p))$. This gives the point 1. and the point 3 (except for "strictly above ") of lemma 19.

To obtain "the strictly above", we notice that along a period the orbit goes through $\Gamma(t_0)$, where there is a "strictly above".

To obtain a symmetric Lagrangian function , we add $\psi_1\circ s$ and we obtain the result. ∎

3.3 End of the proof of theorem 2 and its corollaries

Using Baire's theorem and proposition 18, we obtain that the following property is s-generic: "for every $q_1,q_2\in\mathbb{Q}$ such that $0<q_1<q_2$, there exists a s-symmetric closed 1-form λ on M such that $\mathcal{A}(L+\lambda)$ is one symmetric periodic hyperbolic orbit and such that $\mathcal{A}(L+\lambda)\subset\{(x,p)\in T^*M; H(x,p-\lambda(x))\in]c(L)+q_1,c(L)+q_2[\}$".
This gives directly theorem 2.

Now let L be such a s-generic Lagrangian function and let E be the set of $c>c(L)$ such that $\{H=c\}$ contains one hyperbolic non critical periodic orbit. By the previous property, E is dense, and by its definition it is open. This gives corollary 3.

Now we assume that $s:\mathbb{T}^n\to\mathbb{T}^n$ and $L:TM\to\mathbb{R}$ s-generic satisfy the hypothesis (a), (b). We will again use theorem 2. We have proved that the property: "there exists a dense subset $D(L)$ of $]c(L),+\infty[$ such that, for every $c\in D(L)$, there exists a s-symmetric closed 1-form λ on M such that $\mathcal{A}(L+\lambda)$ is one symmetric periodic hyperbolic orbit contained in $\{(x,p)\in T^*M; H(x,p-\lambda(x))=c\}$"

is s-generic.

Let L be such a s-generic Lagrangian function and let λ be a symmetric closed 1-form such that $\mathcal{A}(L+\lambda)$ is one symmetric non critical periodic hyperbolic orbit Γ. Let $P:\mathbb{T}^n\to\mathbb{T}^n$ be a finite covering of $\mathbb{T}^n$ such that $P^{-1}(\pi^*\circ\Gamma)$ has at least two connected components, $\Gamma_1,\dots,\Gamma_N$. If $\tilde{L}=(L+\lambda)\circ DP$, it is proved in [5] that $c(\tilde{L})=c(L+\lambda)$ and that $\mathcal{A}(\tilde{L})=P^*(\mathcal{A}(L+\lambda))$. Therefore $\mathcal{A}(\tilde{L})$ is not connected. But we know (see [5]) that $\mathcal{N}(\tilde{L})$ is connected and that the ω- and α-limit sets of every element of $\mathcal{N}(\tilde{L})$ is contained in $\mathcal{A}(\tilde{L})$. Therefore $\mathcal{N}(\tilde{L})$ contained at least one orbit which is a heteroclinic orbit Ξ between one periodic Γ_i and one periodic Γ_j (we may have $\Gamma_i=\Gamma_j$). Then $P\circ\Xi$ is a homoclinic orbit to Γ.

4 Appendices

4.1 Appendix A: proof of proposition 5

Let $s : M \to M$ be a C^∞- diffeomorphism which is an involution: $s^2 = Id_M$ such that the set Fix(s) of fixed points of s is a (non-empty) hypersurface $\mathcal{F}$ of M.

Let x be a point of $\mathcal{F}$. Let us prove that $Ds(x)$ is a reflexion. As s in an involution, we have: $(Ds(x))^2 = Id$ and thus $Ds(x)$ is diagonalizable with eigenvalues $+1$ and -1. Moreover, $Ds(x)_{|T_x\mathcal{F}} = Id_{T_x\mathcal{F}}$; then, if there exists one eigenvector for the eigenvalue -1, $Ds(x)$ is a reflexion. Let us prove that there exists such an eigenvector. We work in a chart U near x. Let us consider a sequence (x_n) of point of $M\backslash\mathcal{F}$ converging to x. Then we have: $\forall n \in \mathbb{N}, s(x_n) \neq x_n$. The function s being C^∞, there exists a constant $K > 0$ such that: $\forall y, z \in U, \|s(z) - s(y) - Ds(z)(z - y)\| \leq K\|z - y\|^2$. As $s(x_n) - x_n = s(x_n) - s(s(x_n))$, we deduce:

$$\forall n \in \mathbb{N}, \|s(x_n) - x_n - Ds(x_n)(x_n - s(x_n))\| \leq K\|s(x_n) - x_n\|^2.$$

If we let $u_n = \frac{s(x_n)-x_n}{\|s(x_n)-x_n\|}$, we obtain: $u_n = -Ds(x_n)u_n + o(1)$. Extracting a sequence, we may assume that (u_n) tends to u such that $\|u\| = 1$. Then we have: $u = -Ds(x)u$, i.e. u is an eigenvector for the eigenvalue -1.

Let us now prove that $M\backslash\mathcal{F}$ has at most two connected components: let B be one of the connected components of $M\backslash\mathcal{F}$ and $\mathcal{F}' \subset \mathcal{F}$ its boundary; then $B \cup \mathcal{F}' \cup s(B)$ is a open and closed subset of M, therefore is M; thus $M\backslash\mathcal{F}$ has at most two connected components.

Example : It may happen that $M\backslash\mathcal{F}$ has exactly one connected component: $s : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $s(\theta_1, \theta_2) = (\theta_2, \theta_1)$ is such that $\text{Fix}(s) = \{\theta_1 = \theta_2\}$ and therefore $\mathbb{T}^2\backslash\text{Fix}(s)$ is connected.

4.2 Appendix B: some consequences of the symmetry of L

Let us assume that L is a s-symmetric Lagrangian function : $\forall(x, v) \in TM, L(s(x), -Ds(x)v) = L(x, v)$. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian function associated to L and $\mathcal{L} = \frac{\partial L}{\partial v} : TM \to T^*M$ the associated Legendre map. We introduce the notation :

Notation : $S^* : T^*M \to T^*M$ is the diffeomorphism such that: $S^*(x, p) = (s(x), -p \circ (Ds(x))^{-1})$.
Let us notice that S^* is an antisymplectic diffeomorphism: $\forall(x, p) \in T^*M, \forall(U, V) \in$

$T_{(x,p)}(T^*M), \omega(DS^*(x,p)U, DS^*(x,p)V) = -\omega(U,V)$.

Proposition 21 *With these notations, we have:*

1. $\forall (x,v) \in TM, \mathcal{L}(s(x), -Ds(x)v) = -\mathcal{L}(x,v) \circ (Ds(x))^{-1}$;
2. $H \circ S^* = H$;
3. $\forall t \in \mathbb{R}, \phi^H_{-t} \circ S^* = S^* \circ \phi^H_t$;
4. $u : M \to \mathbb{R}$ *is a c-subsolution if and only is* $-u \circ s$ *is a c-subsolution.*

Proof To obtain the first item, we only have to compute the derivatives in the equality:

$$\forall (x,v) \in TM, L(s(x), -Ds(x)v) = L(x,v).$$

We deduce from this first item that:

$$\forall (x,p) \in T^*M, \mathcal{L}^{-1} \circ S^*(x,p) = (s, -Ds) \circ \mathcal{L}^{-1}(x,p).$$

Therefore, if we write $\mathcal{L}(x,v) = (x,p)$: $H \circ S^*(x,p) = (-p \circ (Ds(x))^{-1})(-DS(x)v) - L(s(x), -Ds(x)v) = pv - L(x,v) = H(x,p)$.

We deduce the third item from the equality $H \circ S^* = H$ and from the fact that S^* is antisymplectic.

To obtain the fourth item, we write the following equivalent sentences:

- $\forall x \in M, H(x, du(x)) \leq c$;
- $\forall x \in M, H(s(x), -du(x) \circ (Ds(x))^{-1}) = H(x, du(x)) \leq c$;
- $\forall x \in M, H(x, -du(s(x)) \circ Ds(x)) = H(x, -du(s^{-1}(x)) \circ (Ds(s^{-1}(x)))^{-1}) \leq c$ because $s = s^{-1}$;
- $\forall x \in M, H(x, -d(u \circ s)(x)) \leq c$.

∎

4.3 Appendix C: on the correspondence between H and L

In this appendix, we recall without proving them some well known results concerning some transformations of the Lagrangian and Hamiltonian functions. We assume that $L : TM \to \mathbb{R}$ is superlinear and strictly convex in the fiber and we name $H : T^*M \to \mathbb{R}$ the associated Hamiltonian function . Then:

1. if $c \in \mathbb{R}$, $L + c$ has the same orbits, critical level and Aubry set as L and the associated Hamiltonian function is $H - c$;
2. if $\psi \in C^\infty(M, \mathbb{R})$, then the Hamiltonian function associated to $L + \psi$ is $H - \psi$;
3. if λ is a closed 1-form on M, the Lagrangian function $L + \lambda : (x, v) \to L(x, v) + \lambda(x)v$ has the same Euler-Lagrange flow as L and the associated Hamiltonian function is: $(x, p) \in T^*M \to H(x, p - \lambda(x))$; moreover, if λ_1 and λ_2 are in the same cohomology class, $L + \lambda_1$ and $L + \lambda_2$ have the same critical value and we have: $\mathcal{A}(L + \lambda_1) = \{(x, p + \lambda_2(x) - \lambda_1(x)); (x, p) \in \mathcal{A}(L + \lambda_2)\}$.

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